An analytic BPHZ theorem for Regularity Structures

Paths to, from and in renormalization - Potsdam

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- **Passage from abstract to concrete:** Encoded by a *model* $Z = (\Pi, \Gamma)$. Here $\Pi = \{\Pi_x\}_{x \in \mathbf{R}^d}$, all of the Π_x's map an indeterminant τ to a concrete element of $\mathcal{S}'(\mathbf{R}^d)$. Π and Γ need to satisfy fairly restrictive algebraic and analytic conditions. $\Pi_x[\tau]$ is the **"homogenous" x -centered incarnation of the space-time distribution represented by the indeterminant τ**

Key Bound: $|(\Pi_x[\tau])(\theta_{x,\lambda})| \lesssim \lambda^{|\tau|}$ uniformly for λ small.

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Positive renormalization counterterms are not designed to kill the divergences that appear in the individual terms of perturbation theory. Instead they enforce the property of *homogeneity*. This property is what allows the fixed point relations for non-perturbative parts of the expansion to be closed.

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- Since we're using an extended regularity structure we can define these maps succinctly.

$$\hat{\Pi}^{\epsilon}_{x}[\tau](\bullet) = \Big(\boldsymbol{\Pi}^{\mathbb{E},\epsilon}[\cdot](0) \otimes \boldsymbol{\Pi}^{\epsilon}[\cdot](\bullet) \otimes \boldsymbol{\Pi}^{\epsilon}[\cdot](x)\Big)(\mathcal{A}_{-} \otimes \mathbf{1} \otimes \mathcal{A}_{+})(\mathbf{1} \otimes \Delta)\hat{\Delta}\tau.$$

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- Our point of departure: We have inserted all the counterterms we think we need but it is not immediate that we have the bound:

$$|(\Pi^{\epsilon}_{x}\tau)(\theta_{\lambda,x})| \lesssim \lambda^{|\tau|} \text{ uniformly for } \epsilon, \lambda \text{ small, almost surely in } \zeta.$$

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    - Generalizes bound found in Hairer-Quastel.

Organizing the sum over scales

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&\leq \sum_{T \in \mathcal{T}} \sum_{\substack{\ell \in \mathcal{L}_T \\ \ell(v_*) \geq \alpha}} \sum_{\mathbf{j} \in t^{-1}[(T,\ell)]} \left| \int \mathrm{d}x_{\mathcal{V}} \; H^{\mathbf{j}}(x_{\mathcal{V}}) \right|
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Want to bound, for a fixed $T\in\mathcal{T}$,

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A rough sketch of how counterterms are organized.

$$\sum_{\mathcal{F} \in \mathbb{F}} \sum_{\mathcal{C} \in \mathfrak{C}_{\mathcal{F}}} \int \mathrm{d}x_{\mathcal{V}} \; H_{\mathcal{C},\mathcal{F}}(x_{\mathcal{V}})$$

Where

$\mathbb{F} =$ collection of all forests of acceptable divergent structures

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$$\sum_{\mathcal{F}\in\overline{\mathbb{F}}}\sum_{\mathcal{C}\subseteq\overline{\mathfrak{C}_{\mathcal{F}}}}\int \mathrm{d}x_{\mathcal{V}}\ H_{\mathcal{C},\mathcal{F}}(x_{\mathcal{V}})$$

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$\overline{\mathbb{F}} =$ collection of all forests of divergent structures

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- Given an edge $e = (e_-, e_+) \in \overline{\mathfrak{C}}$, we want to harvest the positive renormalization corresponding to e if

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$$j_{\{e_+,0\}} > j_{\{e_-,0\}}.$$

- The situation for negative renormalizations is more complicated. For now we forget about positive renormalizations. Define "projection operator" $P^{\mathbf{j}} : \overline{\mathbb{F}} \mapsto \overline{\mathbb{F}}$ by setting

$$P^{\mathbf{j}}[\mathcal{F}] = \left\{ T \in \mathcal{F} : \begin{array}{c} \text{the largest scale index among the} \\ \text{external edges (mod } \mathcal{F}\text{) of } T \\ \textbf{is greater than or equal to} \\ \text{the minimum scale index among the} \\ \text{internal edges (mod } \mathcal{F}\text{) of } T. \end{array} \right\}$$

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$$\mathfrak{M} \;:=\; \left\{ \mathbb{M} \subseteq \overline{\mathbb{F}} : \begin{array}{c} \exists \mathcal{S} \in \overline{\mathbb{F}},\ \mathbf{j} \in \mathcal{N} \text{ such that} \\ P^{\mathbf{j}}[\mathcal{S}] \;=\; \mathcal{S} \text{ and } \mathbb{M} = (P^{\mathbf{j}})^{-1}[\mathcal{S}] \end{array} \right\}$$

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- $\mathfrak{M}$ is a collection of "intervals" of the poset $\overline{\mathbb{F}}$. Idea: Summing $\mathcal{F} \in \mathbb{M}$ for fixed $\mathbb{M} \in \mathfrak{M}$ renormalizes the divergences that appear for coalesence trees determined by $\mathbf{j} \in \mathcal{N}_{\mathbb{M}}$ on subsets of graph vertices where the elements of $Min(\mathbb{M})$ are contracted:

$$\sum_{\mathbf{j} \in \mathcal{N}_{\mathbb{M}}} \left| \sum_{\mathcal{F} \in \mathbb{M}} \int \mathrm{d}x_{\mathcal{V}}\ H^{\mathbf{j}}_{\varnothing,\mathcal{F}}(x_{\mathcal{V}}) \right| < \infty$$

- $P^{\mathbf{j}}$ is called a forest projection because it satisfies $(P^{\mathbf{j}})^2 = P^{\mathbf{j}}$. Define

$$\mathfrak{M} := \left\{ \mathbb{M} \subseteq \overline{\mathbb{F}} : \begin{array}{c} \exists \mathcal{S} \in \overline{\mathbb{F}},\ \mathbf{j} \in \mathcal{N} \text{ such that} \\ P^{\mathbf{j}}[\mathcal{S}] = \mathcal{S} \text{ and } \mathbb{M} = (P^{\mathbf{j}})^{-1}[\mathcal{S}] \end{array} \right\}$$

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- Similar to overlapping divergences, the positive and negative renormalizations we need simultaneously are non-overlapping in phase-space so both can be harvested at the same time. By using a hierarchy of distributional norms we can deal with nested divergences through an inductive procedure.

Theorem (C. - Hairer)

For subcritical perturbations of the heat equation driven by noise no worse than space-time white noise (with good cumulant bounds) there is a renormalization scheme for the canonical model yielding renormalized models $\hat{Z}^{\epsilon}$ *which for every* τ, *and each* $p \in \mathbf{N}$ *satisfy the bound*

$$\mathbf{E}\Big[\Big|(\hat{\Pi}_0^{\epsilon}\tau)(\theta_{0,\lambda})\Big|^p\Big] \lesssim \lambda^{p(|\tau|+\kappa)} \text{ uniformly for } \lambda, \epsilon \text{ small.}$$

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Thanks for listening!