Anderson Hamiltonian with white noise potential

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Joint work with R.Allez

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Construction and Statistical properties

Goal 1: Define the stochastic random Schrödinger operator on $[0,1]^d$ for $d = 1, 2, 3$

$$\mathscr{H} := -\Delta + \eta$$

where η is a Gaussian white noise; for $x, y \in \mathbb{R}^d$,

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Goal 2: It is a self-adjoint operator; What about its statistical spectral properties?

Motivation and related question

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- Anderson-Localization?

Schrödinger Operator with smooth potential

Smooth potential

Let V a $L^\infty(\mathbb{T}_L^d)$ function and we define the operator

$$H^V f = -\Delta f + fV$$

for all $f \in H^2(\mathbb{T}_L^d)$. H^V is a self-adjoint unbounded operator of $L^2(\mathbb{T}_L^d)$ with domain $H^2(\mathbb{T}_L^d)$.

Spectral analysis of H^V

1. The spectrum of H^V coincide with the punctual spectrum and

$$S_p(H^V) = \{\Lambda_1^c(V) \leq \Lambda_2^c(V) \leq ... \leq \Lambda_n^c(V)...\}$$

   without accumulation point and such that $\Lambda_n(V) \to +\infty$
2. $L^2(\mathbb{T}_L^2) = \bigoplus_{n\in\mathbb{N}} Ker(\Lambda_n - H^V)$
3. $\Lambda_n(V) = \min_{F\subset H^2(\mathbb{T}^2)} \max_{f\in F;\|f\|_{L^2}=1} \langle H^V f, f\rangle$
4. $|\Lambda_n(V) - \Lambda_n(\tilde{V})| \leq \|V - \tilde{V}\|_\infty$

One dimensional case

With Periodic boundary conditions,

$$-\frac{d^2}{dx^2} + b'_t, \quad (b \in \mathscr{C}^{1/2-})$$

on the space $H^1([-L, L], \mathbb{R})$, $L > 0$ such that

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on the space $H^1([-L, L], \mathbb{R})$, $L > 0$ such that
If $f \in H^1([-L, L], \mathbb{R})$,

$$f(t)b_t' \in H^{-1/2-}$$

is a well defined as a distribution.

Why H^1?

Let (f, λ) a solution of the eigenvalue equation then

$$-\frac{d^2}{dx^2} f = \lambda f + f b'$$

then we can expect that the regularity of f is at least the regularity of b' $+2$ then $f \in \mathscr{C}^{3/2-} \subseteq H^{3/2-}$

The random spectrum

Theorem (Fukushima, Nakao (1977))

the following inequality hold

$$C_1(b)\|f\|_{H^1} \leq \langle Hf, f\rangle_{L^2} + \gamma\|f\|_2^2 \leq C_2(b)\|f\|_{H^1}$$

Then $\mathscr{H}$ *admit a unique self-adjoint extension. The random spectrum of* $\mathscr{H}$ *is almost surly pure point formed by a sequence* Λ_k. *Furthermore*

$$\Lambda_1 < \Lambda_2 < \Lambda_3 < \dots$$

Moreover $L^2 = \bigoplus_k \ker(\Lambda_k - \mathscr{H})$

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Schrödinger operator with white noise potential

Goal

We want to define and study the follwoing random Schrödinger operator :

$$\mathscr{H} = -\Delta + \eta, \quad d = 2$$

as an unbounded operator of $L^2(\mathbb{T}_L^2)$.

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Problem

1. η is a Schwartz distribution and not a function. $\eta \in \mathscr{C}^{-1-\varepsilon}(\mathbb{T}_L^2)$ for all $\varepsilon > 0$ and not bettre
2. If φ a smooth function then the product $\varphi\eta$ is well defined as a distribution !
3. If (f, Λ) is an eigenfunction/eigenvalue of $\mathscr{H}$:

   $$-\Delta f = f\eta + \Lambda f$$

   regularity of f = (regularity of η) $+ 2 = 1 - \varepsilon \implies$ the product $f\eta$ is ill-defined.

Non homogenous Besov space

1. for $j > 0$, Δ_j the projection on the Fourier mode of size 2^j, Δ_{-1} the projection on the Fourier mode less than 2.
2. $f \in \mathscr{S}'(\mathbb{T}^d)$, we say that $f \in \mathscr{B}^{\alpha}_{p,q}$ if $\left(2^{j\alpha}\|\Delta_j f\|_{L^p(\mathbb{T}^d)}\right)_{j\geq -1} \in l_q([-1,+\infty))$, moreover when $p = q = 2$ then it coincide with the Sobolev space H^α
3. If $\alpha \in (0,1)$ then $\mathscr{C}^\alpha = \mathscr{B}^{\alpha}_{\infty,\infty}$ coincide with the space of α-Hölder functions.
4. $f \in \mathscr{B}^{\gamma}_{p,p}$, $g \in \mathscr{B}^{\alpha}_{\infty,\infty}$ then the Paraproduct term

   $$f \prec g = \sum_{-1\leq i<j-1} \Delta_i f \Delta_j g$$

   is always well defined and satisfy that $f \prec g \in \mathscr{B}^{\min(\alpha,\alpha+\gamma-)}_{p,p}$ (with good continuity bound).

   $$fg = f \prec g + f \circ g + f \succ g$$

   with $f \succ g = g \prec f$ and $f \circ g = \sum_{|i-j|\leq 1} \Delta_i f \Delta_j g$ is well defined if $\alpha + \gamma > 0$ and in this case it lie in $\mathscr{B}^{\alpha+\gamma}_{p,p}$.
5. The white noise η satisfy that almost surely $\eta \in \mathscr{C}^{-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)$ for all $\varepsilon > 0$.

Paracontrolled distribution and Domain of the operator

Decomposition of the eigenfunction

If (f, λ) is a formal solution of the eigenvalue problem associated to $\mathscr{H}$ then

$$-\Delta f = -f\eta + \lambda f = \underbrace{f \prec \eta}_{H^{-1-\varepsilon}} - \underbrace{f \succ \eta + f \circ \eta - \Lambda f}_{H^{-\varepsilon}}$$

$\Rightarrow$

$$f = -\underbrace{(1-\Delta)^{-1}(f \prec \eta)}_{1-\varepsilon} - \underbrace{(1-\Delta)^{-1}\left(f \succ \eta + f \circ \eta - \Lambda f + f\right)}_{2-\varepsilon}$$

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Domain of the operator

Define $X := -(1-\Delta)^{-1}\eta$

$$\mathscr{D}_\eta = \left\{ f \in H^{1-\varepsilon}, \quad f^\sharp := f - f \prec X \in H^{2-\varepsilon} \right\}$$

equipped with the scalar product $\langle f, g\rangle_{\mathscr{D}_\eta} = \langle f, g\rangle_{H^{1-\varepsilon}} + \langle f^\sharp, g^\sharp\rangle_{H^{2-\varepsilon}}$

Extension of the product

Resonating term

$f \in \mathscr{D}_\eta$

$$f \circ \eta = (f \prec X) \circ \eta + \underbrace{f^\sharp \circ \eta}_{-1-\varepsilon+2-\varepsilon=1-2\varepsilon>0}$$

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Lemma (M.Gubinelli,P.Imkeller, N.Perkowski)

$(f, g, h) \in H^\gamma \times \mathscr{C}^\alpha \times \mathscr{C}^\beta$ *with* $\alpha + \beta + \gamma > 0$ *then the trilinear operator*

$$\mathscr{R}(f, g, h) = (f \prec g) \circ h - f(g \circ h)$$

is well-defined and it lie in $H^{\alpha+\beta+\gamma-}$

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Extended product

$$f \circ \eta = f(\eta \circ X) + \underbrace{\mathscr{R}(f, X, \eta) + f^\sharp \circ \eta}_{\text{Well-defined}}$$

Definition of the operator $\mathscr{H}$

Given $(\eta, \Xi) \in \mathscr{C}^{-1-\varepsilon} \times \mathscr{C}^{-\varepsilon}$ and $f \in \mathscr{D}_\eta$ we define the Schrödinger operator by :

$$\mathscr{H}(\eta, \Xi)f = -\Delta f + (f \star \eta) \in H^{-1-\varepsilon}$$

with

$$f \star \eta = f \prec \eta + f \succ \eta + f\Xi + \mathscr{R}(f, X, \eta) + f^\sharp \circ \eta$$

and we can see that when η smooth function and that $\Xi := \eta \circ X$ then $\mathscr{H}(\eta, \Xi)f = -\Delta f + f\eta$.

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Remark

$$\mathscr{H}(\eta, \Xi) f = -\Delta f + \underbrace{f \prec \eta}_{-1-\varepsilon} + f \succ \eta + f\Xi + \mathscr{R}(f, \eta, X) + f^\sharp \circ \eta$$

$$= -\Delta f^\sharp - 2\nabla f \prec \nabla X - \Delta f \prec \eta + f \succ \eta + f\Xi + \mathscr{R}(f, \eta, X) + f^\sharp \circ \xi \in H^{-2\varepsilon}$$

Then $\mathscr{H} f$ is not yet in $L^2(\mathbb{T}_L^2)$. Pushing further the expansion we can construct a space $\mathscr{D}_{\eta,\Xi} \subset \mathscr{D}_\eta$

Resolvent

Resolvent

There exist $a^\star = a^\star(||\Xi_2|| + ||\xi||^2) > 0$ such that for all $a \geq a^\star$ and $g \in L^2(\mathbb{T}_L^2)$ the equation :

$$(\mathscr{H}(\eta, \Xi) + a)f = g$$

admit a unique solution $f = R_a g \in \mathscr{D}_{\eta,\Xi}$, $R_a : L^2(\mathbb{T}_L^2) \to \mathscr{D}_{\eta,\Xi}$ is a bounded operator and if we see $R_a : L^2(\mathbb{T}_L^2) \to L^2(\mathbb{T}_L^2)$ is compact self-adjoint operator.

The proof of this result is based on a fixed point argument in the space $\mathscr{D}_\Xi$.

$$\Gamma(f) = (-\Delta + a)^{-1}(f \star \eta + g)$$

The same technique can be applied to solve and get the convergence result for Parabolic SPDE presented previously.

Main analytical result

Theorem

Given $\alpha \in (-\frac{4}{3}, -1)$ there exist a Banach space $\mathscr{X}^{\alpha} \subset \mathscr{C}^{\alpha}(\mathbb{T}_L^2) \times \mathscr{C}^{2\alpha+2}(\mathbb{T}_L^2)$, such that for all $R\Xi = (\eta, \Xi) \in \mathscr{X}^{\alpha}$ there existe a Hilbert space $\mathscr{D}_{\Xi} \subset L^2(\mathbb{T}^2)$ (dense in L^2) and a unique self-adjoint operator $\mathscr{H}(\eta, \Xi) : \mathscr{D}_{\Xi} \to L^2(\mathbb{T}^2)$ such that

1. If η is a smooth function then we can choose Ξ such that if for all $c \in \mathbb{R}$
$$\mathscr{D}_{(\eta,\Xi+c)} = H^2(\mathbb{T}^2), \quad \mathscr{H}(\eta, \Xi + c)f = -\Delta f + f(\eta + c)$$
for all $f \in H^2(\mathbb{T}^2)$
2. The spectrum of $\mathscr{H}(\eta, \Xi)$ is real, discrete without accumulation point and formed by sequence $(\Lambda_n(\Xi))_{n\in\mathbb{N}^*}$ which satisfy $\Lambda_n(\Xi) \to +\infty$,
$$\Lambda_1(R\Xi) \leq \Lambda_2(R\Xi) \leq \ldots \leq \Lambda_n(R\Xi)$$
and $\dim(\Lambda_n(R\Xi) - \mathscr{H}(\eta, \Xi)) = 1$. Moreover $L^2(\mathbb{T}^2) = \bigoplus_n \ker(\Lambda_n(\Xi) - \mathscr{H}(\Xi))$
3. Λ_n satisfy a min-max principle.
4. For each $n \in \mathbb{N}$, $\Xi \to \Lambda_n(\Xi)$ is locally-Lipschitz. More precisely
$$|\Lambda_n(\eta, \Xi) - \Lambda_n(\tilde{\eta}, \tilde{\Xi})| \leq$$
$$Cn\left(1 + n^{\frac{2\gamma-\alpha}{\alpha+2}} + (1 + \Lambda_n(0))^{2\gamma}\right)^2 \|R\Xi - \tilde{R}\Xi\|_{\mathscr{X}^\alpha}(1 + \|R\Xi\|_{\mathscr{X}^\alpha} + \|R\Xi\|_{\mathscr{X}^\alpha})^M$$
for all $\gamma < \alpha + 2$, $\Xi, \tilde{\Xi} \in \mathscr{X}^{\alpha}$ and where $\Lambda_n(0)$ is the n-lowest eigenvalue of the Laplaician, C and M are two positive constant which depend only on γ and α .
5. $R\Xi \to \mathscr{H}(R\Xi)$ is continuous in resolvent sense.

Discontinuity

Example

Take $z = (1,1)$, $v \in \mathscr{C}^\infty(\mathbb{T}^2)$ and introduce $v_n(x) = n\cos(2n\pi\langle z, x\rangle)v(x)$ then we can check that

$$(v_n, -v_n \circ (1-\Delta)^{-1} v_n) \to^{n\to+\infty} (0, -v^2)$$

in $\mathscr{C}^\alpha \times \mathscr{C}^{2\alpha+2}$ for all $\alpha < -1$. Which say in particularly that the k-lowest eigenvalue of $-\Delta + v_n$ converge to the k-lowest eigenvalue of $-\Delta - v^2$.

Come-back to the white noise

Theorem

Given $\alpha < -1$, η the white noise and η^ε a mollification of it and $\Xi^\varepsilon = \eta^\varepsilon \circ X^\varepsilon$ then there exist $R\Xi^{wn} = (\eta, \Xi^{wn}) \in \mathscr{X}^\alpha$ and a constant $c_\varepsilon \to^{\varepsilon \to 0} +\infty$ such that the follwoing convergence hold

$$(\eta^\varepsilon, \Xi^\varepsilon + c_\varepsilon) \to^{\varepsilon \to 0} (\eta, \Xi^{wn})$$

in $L^p(\Omega, \mathscr{X}^\alpha)$ for all $p > 0$ and almost surly in $\mathscr{X}^\alpha$. Moreover $R\Xi^{wn}$ does not depend on the choice of the mollifier and we have :

$$c_\varepsilon = \frac{1}{2\pi} \log(\frac{1}{\varepsilon}) + O_L(1)$$

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Theorem

Denote by

$$\Lambda_1^\varepsilon \leq \Lambda_2^\varepsilon \leq \Lambda_3^\varepsilon \leq \cdots$$

the eigenvalues of the operator $\mathscr{H}_\varepsilon = -\Delta + \eta^\varepsilon$. Then, for any $n \in \mathbb{N}$, almost surely,

$$\Lambda_n^\varepsilon + c_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \Lambda_n(R\Xi^{wn}).$$

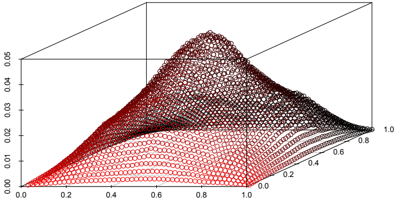
Moreover there exist two positive constant C_1 and C_2 such that

$$e^{C_2 x} \leq \mathbb{P}(\Lambda_1(R\Xi^{wn}) \leq x) \leq e^{C_1 x}$$

when x goes to $-\infty$. Besides we have

$$\sup_L \mathbb{E}\left[\left|\frac{\Lambda_1(R\Xi^{wn})}{\log L}\right|^p\right] < +\infty$$

Eigenvector approximation



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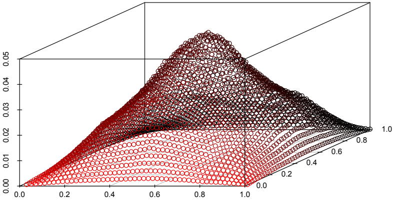

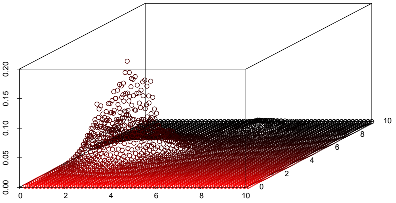

Heuristic

Scaling argument

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which imply

$$\Lambda_n(\eta) = \frac{1}{r^2}\Lambda_n(r^2\eta(r\cdot)) \equiv^{\text{Loi}} \frac{1}{r^2}\Lambda_n(r\tilde{\eta})$$

η white noise on $\mathbb{T}^2_L$ and $\tilde{\eta}$ is a white noise on $\mathbb{T}^2_{\frac{L}{r}}$.

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$$\|\xi\|_{\mathscr{C}^\alpha(\mathbb{T}^2_{\frac{1}{\delta}})} \leq \underbrace{C}_{\text{deterministic constant}} \sqrt{\log\frac{1}{\delta}} + \underbrace{A}_{\text{Random constant exponentially integrable}}$$

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$r = \frac{1}{\sqrt{\log L}}$ donc $r\|\tilde{\eta}\|_{\mathscr{C}^\alpha(\mathbb{T}^2_{L\sqrt{\log L}})} \lesssim 1$ which in particularly imply that:

$$|\Lambda_n(\eta)| \lesssim \frac{1}{r^2} = \log L\text{''}$$

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Remark

The exponential tail are obtained by observing that

$$\mathbb{P}(\Lambda_1(\eta) \leq -x) = \mathbb{P}\left(\Lambda_1(\frac{1}{\sqrt{-x}}\tilde{\eta}) \leq -1\right)$$

Large deviation event.

Infinite volume case

If we want to construct

$$\mathscr{H} = -\Delta f + f\eta$$

where η is a white noise on $\mathbb{R}^2$.

problem

η is not more a $\mathscr{C}^{-1-}$ distribution but

$$\|\eta\|_{\mathscr{C}^{\alpha},w} = \sup_i 2^{i\alpha}\|w\Delta_i f\|_{L^\infty} < +\infty$$

where $\alpha < -1$ and $w : \mathbb{R}^2 \to (0,+\infty)$ smooth such that $w(x) \sim^{|x|\to+\infty} \frac{1}{\sqrt{\log|x|}}$

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Domain

$\eta \in \mathscr{C}^{\alpha,w}$

$$\mathscr{D}_{\eta,w} = \left\{ f \in H^{\alpha+2,\frac{1}{w}}, f - f \prec X \in H^{2\alpha+4,\frac{1}{w}} \right\}$$

Theorem: Pasteur

Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space and A a random operator $\omega \to A(\omega)$ measurable almost surly self adjoint and ergodic in the following sense

$$A(\tau_y \omega) = U_y^\star A(\omega) U_y$$

for an ergodic family $\tau_y : \Omega \to \Omega$ ergodic family and $(U_y)_{y\in I}$ family of unitary operator. Then the spectrum of H is deterministic (equal almost surly to a closed set of $\mathbb{R}$).

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If we want to construct

$$\mathscr{H} = -\Delta f + f\eta$$

where η is a white noise on $\mathbb{R}^2$.

problem

η is not more a $\mathscr{C}^{-1-}$ distribution but

$$\|\eta\|_{\mathscr{C}^{\alpha,w}} = \sup_i 2^{i\alpha}\|w\Delta_i f\|_{L^\infty} < +\infty$$

where $\alpha < -1$ and $w : \mathbb{R}^2 \to (0,+\infty)$ smooth such that $w(x) \sim^{|x|\to+\infty} \frac{1}{\sqrt{\log|x|}}$

Domain

$\eta \in \mathscr{C}^{\alpha,w}$

$$\mathscr{D}_{\eta,w} = \left\{ f \in H^{\alpha+2,\frac{1}{w}}, f - f \prec X \in H^{2\alpha+4,\frac{1}{w}} \right\}$$

Theorem: Pasteur

Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space and A a random operator $\omega \to A(\omega)$ measurable almost surly self adjoint and ergodic in the following sense

$$A(\tau_y\omega) = U_y^\star A(\omega) U_y$$

for an ergodic family $\tau_y : \Omega \to \Omega$ ergodic family and $(U_y)_{y\in I}$ family of unitary operator. Then the spectrum of H is deterministic (equal almost surly to a closed set of $\mathbb{R}$). In our case $\mathbb{P} =$ law of the white noise, $\Omega = \mathscr{S}'(\mathbb{R}^2)$, $\tau_y\omega = \omega(\cdot - y)$ and $U_y f(x) = f(x+y)$.

Thank you