Systems of Dyson-Schwinger equations with several coupling constants

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Feynman graphs

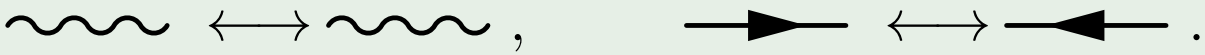
A theory of Feynman graphs $\mathcal{T}$ is given by:

- A set $\mathcal{HE}$ of types of half-edges, with an incidence rule, that is to say an involutive map $\iota : \mathcal{HE} \longrightarrow \mathcal{HE}$.
- A set $\mathcal{V}$ of vertex types, that is to say a set of finite multisets (in other words finite unordered sequences) of elements of $\mathcal{HE}$, of cardinality at least 3.

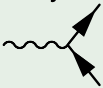
The edges of $\mathcal{T}$ are the multisets $\{t, \iota(t)\}$, where t is an element of $\mathcal{HE}$. The set of edges of $\mathcal{T}$ is denoted by $\mathcal{E}$.

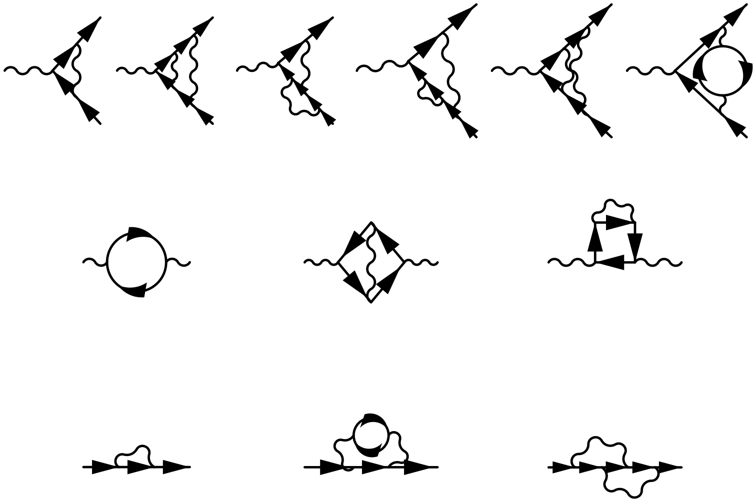
QED

- $\mathcal{HE}_{QED} = \{\sim\!\sim\!\sim, \longrightarrow, \longleftarrow\}$.
- Incidence rule:



- Edges: $\sim\!\sim\!\sim$ and $\longrightarrow$.
- Only one vertex type:

 $= \{\sim\!\sim\!\sim, \longrightarrow, \longleftarrow\}$.



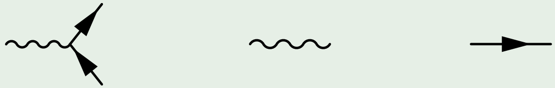
External structure

The external of a Feynman graph in $\mathcal{FG}_{\mathcal{T}}$ is the multiset of its external half-edges.

We only allow Feynman graphs such that the external structure is an edge or a vertex type of the theory $\mathcal{T}$.

In QED

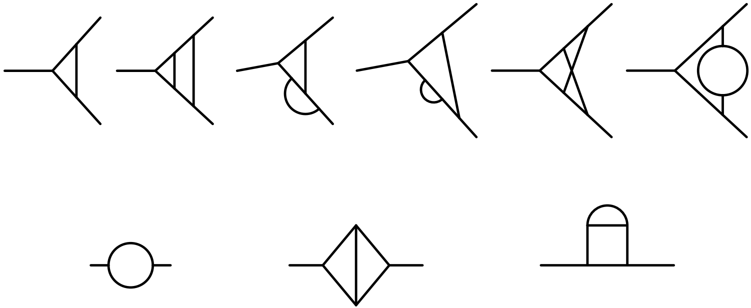
Three possible external structures:



φ^n, $n \geq 3$

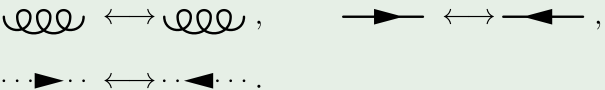
- $\mathcal{E}_{\varphi^n} = \{$———$\}$.
- One edge, denoted by ——— .
- Only one vertex type, which is the multiset formed by n copies of ——— .

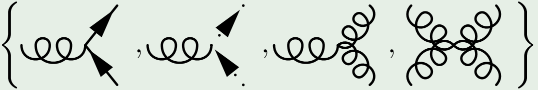
For $n = 3$:

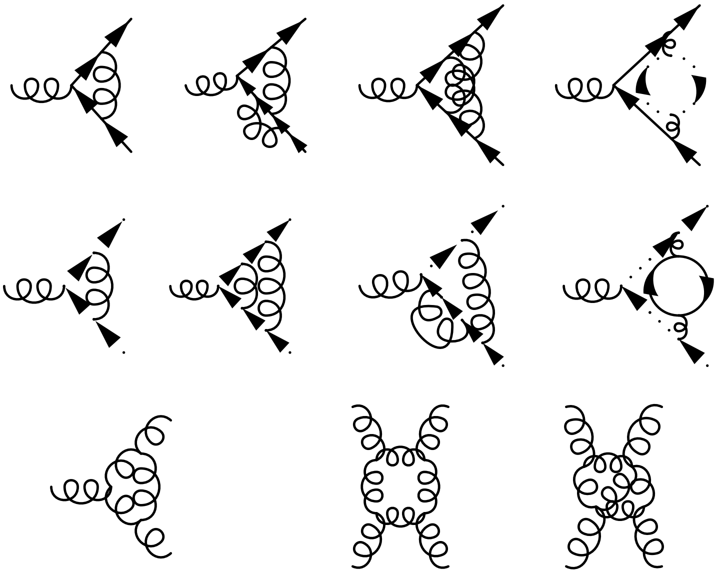


QCD

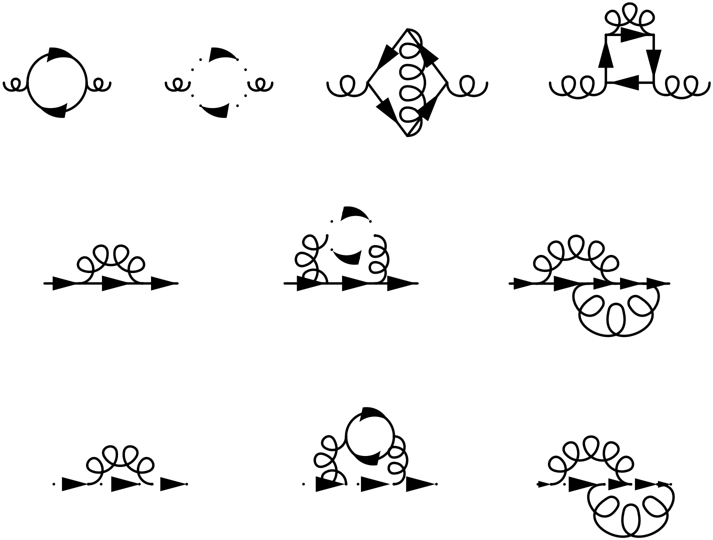
- $\mathcal{HE}_{QCD} = \{$  $\}$.
- Incidence rule:



- Three edges,  (gluon),  (fermion) and  (ghost).
- $\mathcal{V}_{QCD} =$ .



Definition and examples



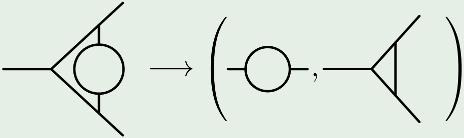
Loop number

The loop number of a Feynman graph G is:

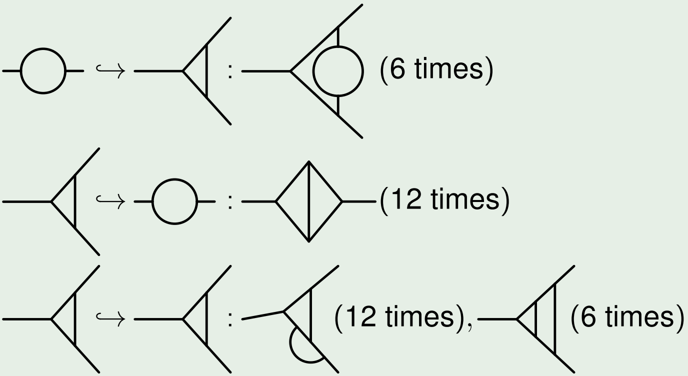
$$\ell(G) = \sharp\{\text{internal edges of } G\} - \sharp\{\text{vertices of } G\} + \sharp\{\text{connected components of } G\}.$$

As we only consider $1PI$ Feynman graphs, for all $G \neq \emptyset$, $\ell(G) \geq 1$.

Extraction-contraction of a subgraph



Insertion



The Connes-Kreimer bialgebra of Feynman graph of a given theory $\mathcal{T}$ is denoted by $\mathcal{H}_{\mathcal{FG}(\mathcal{T})}$.

- A basis of $\mathcal{H}_{\mathcal{FG}(\mathcal{T})}$ is the set of all Feynman graphs of the theory.
- The product is the disjoint union.
- The unit is the empty Feynman graph.
- Coproduct : for any Feynman graph G,

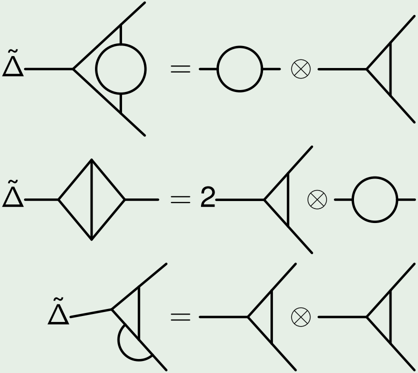
$$\Delta(G) = \sum_{\gamma \subseteq G} \gamma \otimes G/\gamma.$$

Proposition

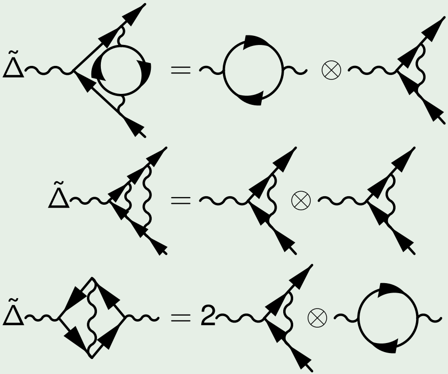
The bialgebra $\mathcal{H}_{\mathcal{FG}(\mathcal{T})}$ is $\mathbb{N}$-graded by the number of loops.

We put $\tilde{\Delta}(x) = \Delta(x) - x \otimes 1 + 1 \otimes x$.

In φ^3

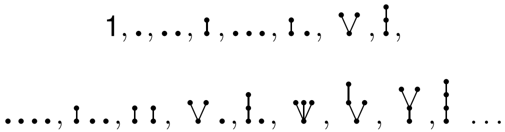


In QED



The Connes-Kreimer bialgebra of rooted trees is denoted by $\mathcal{H}_{CK}$.

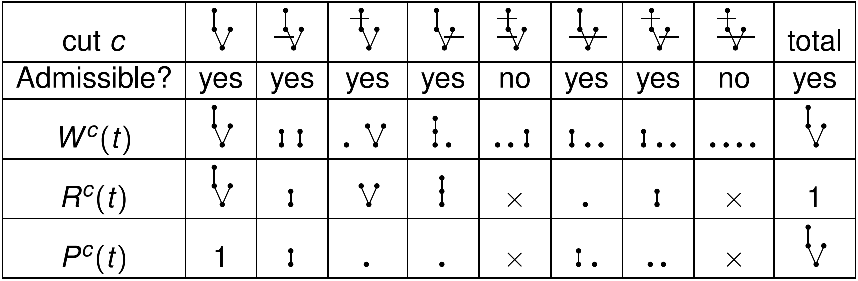
- The set of rooted forests is a basis of $\mathcal{H}_{PR}$:

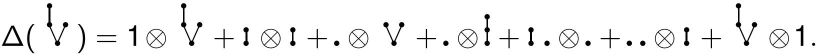


- The product is the disjoint union of forests.

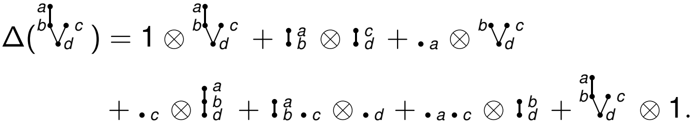
- The coproduct is given by *admissible cuts*:

$$\Delta(t) = \sum_{c \text{ admissible cut}} R^c(t) \otimes P^c(t).$$





Decorated version: choose a set D of decorations. In $\mathcal{H}_{CK}^{D}$, the vertices of rooted trees are decorated by elements of D.



Proposition

We choose a weight for each decoration $d \in D$. This induces a graduation of the bialgebra $\mathcal{H}_{CK}^{D}$.

For each external structure (vertex or edge) i, we consider

$$X_i = \sum_{G \in \mathcal{FG}(\mathcal{T})_i} \alpha^{\ell(G)} s_G G,$$

where:

- $\mathcal{FG}(\mathcal{T})_i$ is the set of connected Feynman graphs of external structure i.
- s_G is a symmetry factor.
- α is an indeterminate (the coupling constant).

These elements lives in a completion of $\mathcal{H}_{\mathcal{FG}(\mathcal{T})}$.

We put:

$$X_i = \sum_{n \geq 1} \alpha^n X_i(n).$$

$X_i(n)$ is a span of Feynman graphs of external structure i with n loops.

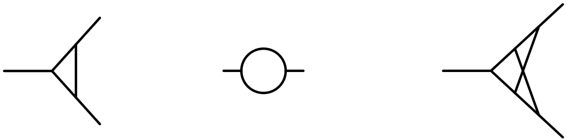
Questions

1. How to inductively describe the elements $X_i(n)$?
2. Is the subalgebra generated by the $X_i(n)$ a subbialgebra of $\mathcal{H}_{\mathcal{FG}(\mathcal{T})}$?
3. If it is a subbialgebra, what can be said on it?
4. If it is not a subbialgebra, what can be done?

A graph G is primitive if it has no proper subgraphs:

$$\Delta(G) = G \otimes 1 + 1 \otimes G.$$

For example, in ϕ^3, the following graphs are primitive:



Any Feynman graph can be obtained by insertion of a graph in a primitive Feynman graph.

Insertion operators

For any primitive Feynman graph G, for any graph γ, $B_G(\gamma)$ is the average of the insertions of γ in G.
Note that is not always defined.

In ϕ^3, two possible external structures, vertex v or edge e.

$$X_v = \sum_{\substack{G \text{ primitive graph} \\ \text{of external structure } v}} \alpha^{\ell(G)} B_G \left(\frac{(1+X_v)^{|Vert(G)|}}{(1-X_e)^{|Int(G)|}} \right)$$

$$X_e = \sum_{\substack{G \text{ primitive graph} \\ \text{of external structure } e}} \alpha^{\ell(G)} B_G \left(\frac{(1+X_v)^{|Vert(G)|}}{(1-X_e)^{|Int(G)|}} \right)$$

In ϕ^3, two possible external structures, vertex 1 or edge 2.

$$X_1 = \sum_{\substack{G \text{ primitive graph} \\ \text{of external structure } 1}} \alpha^{\ell(G)} B_G \left(\frac{(1+X_1)^{|Vert(G)|}}{(1-X_2)^{|Int(G)|}} \right)$$

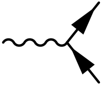
$$X_2 = \sum_{\substack{G \text{ primitive graph} \\ \text{of external structure } 2}} \alpha^{\ell(G)} B_G \left(\frac{(1+X_1)^{|Vert(G)|}}{(1-X_2)^{|Int(G)|}} \right)$$

In ϕ^3, two possible external structures, vertex 1 or edge 2.

$$X_1 = \sum_{k\geq 1} \alpha^k \sum_{\substack{G \text{ primitive graph} \\ \text{of external structure 1} \\ \text{with } k \text{ loops}}} B_G\left(\frac{(1+X_1)^{3k}}{(1-X_2)^{2k-1}}\right)$$

$$X_2 = \sum_{k\geq 1} \alpha^k \sum_{\substack{G \text{ primitive graph} \\ \text{of external structure 2} \\ \text{with } k \text{ loops}}} B_G\left(\frac{(1+X_1)^{3k}}{(1-X_2)^{3k-1}}\right)$$

In QED, three possible external structures:

$1 =$  $2 =$  $3 =$  .

$$X_1 = \sum_{k \geq 1} \alpha^k \sum_{G \in P_1(k)} B_G \left(\frac{(1+X_1)^{2k+1}}{(1-X_2)^k (1-X_3)^{2k}} \right),$$

$$X_2 = \sum_{k \geq 1} \alpha^k \sum_{G \in P_2(k)} B_G \left(\frac{(1+X_1)^{2k}}{(1-X_2)^{k-1} (1-X_3)^{2k}} \right),$$

$$X_3 = \sum_{k \geq 1} \alpha^k \sum_{G \in P_3(k)} B_G \left(\frac{(1+X_1)^{2k}}{(1-X_2)^k (1-X_3)^{2k-1}} \right).$$

Generally:

- The vertex types of $\mathcal{T}$ are indexed by $1, \ldots, k$.
- The edges of $\mathcal{T}$ are indexed by $k+1, \ldots, k+l = M$.

For any Feynman graph G:

- $v_i(G)$ is the number if vertices of G of the i-th vertex type.
- $e_j(G)$ is the number if internal edges of G of the j-th type.

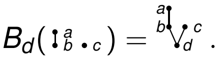
Dyson-Schwinger system $(S_{\mathcal{T}})$ associated to $\mathcal{T}$

if $1 \leq i \leq k+l$:

$$X_i = \sum_{G \in P_i} \alpha^{\ell(G)} B_G \left(\prod_{j=1}^{k} (1+X_j)^{v_i(G)} \prod_{j=k+1}^{k+l} (1-X_j)^{-e_j(G)} \right).$$

Grafting operators

In $\mathcal{H}_{CK}^{D}$, if $d \in D$ and F is a forest, $B_d(F)$ is the tree obtained by grafting the trees of F on a common root decorated by d.



Dyson-Schwinger systems on decorated rooted trees

$D = D_1 \sqcup \ldots \sqcup D_M$, $f_d \in \mathbb{K}[[x_1, \ldots, x_M]]$ for all $d \in D$. Associated system: for all $i \in [M]$,

$$Y_i = \sum_{d \in D_i} \alpha^{weight(d)} B_d(f_d(Y_1, \ldots, Y_M)).$$

Such a system has a unique solution $Y = (Y_1, \ldots, Y_M)$, living in a completion of $\mathcal{H}_{CK}^D$.

System associated to a theory of Feynman graph $\mathcal{T}$:

1. D is the set of primitive Feynman graphs of $\mathcal{T}$.
2. For all $1 \leq i \leq M$, D_i is the set of primitive Feynman graphs of external structure i.

If $1 \leq i \leq M$:

$$Y_i = \sum_{d \in D_i} \alpha^{weight(d)} B_d \left(\prod_{j=1}^{k} (1 + Y_j)^{v_i(d)} \prod_{j=k+1}^{k+l} (1 - Y_j)^{-e_j(d)} \right).$$

From trees to Feynman graphs

Let $\mathcal{T}$ be a theory of Feynman graphs and for all $d \in D$, let G_d be a primitive Feynman graph. There exists a subspace H of $\mathcal{H}_{CK}^D$ and $\phi : H \longrightarrow \mathcal{H}_{\mathcal{FG}(\mathcal{T})}$, compatible with the product and the coproduct, such that for all $d \in D$, $\phi \circ B_d = B_{G_d} \circ \phi$.

In the case where D is the set of primitive Feynman graphs of $\mathcal{T}$, ϕ is injective and for all $1 \leq i \leq M$, $\phi(Y_i) = X_i$.

Proposition

The subalgebra generated by the components of $Y_1, \ldots, Y_M$ is a subbialgebra of $\mathcal{H}_{CK}^D$ if, and only if, the subalgebra generated by the components of $X_1, \ldots, X_M$ is a subbialgebra of $\mathcal{H}_{\mathcal{FG}(\mathcal{T})}$.

Let (S) be a Dyson-Schwinger system in $\mathcal{H}_{CK}^{D}$.

Questions

1. Is the subalgebra generated by the components of $Y_1, \ldots, Y_n$ a subbialgebra of $\mathcal{H}_{CK}^{D}$?
2. If it is a subbialgebra, what can be said on it?
3. If it is not a subbialgebra, what can be done?

In the case where there is a single grafting operator in each equation (restricting to primitive Feynman graphs with one loop only):

1. A classification of the systems giving a subbialgebra is done:
    1. Two main families of systems.
    2. Four operations on these systems (rescaling, concatenation, dilatation, extension).
2. For such a system, there exists a unique extension to a system with an arbitrary number of grafting operators per equation.
3. The description of the structure of the associated subbialgebra is done in terms of a Lie algebra and a group.

Problem

The system for φ^n and for *QED* is such a system.
This is not the case for QCD.

Solution

Refine the graduation by the number of loops. This $\mathbb{N}$-graduation should be replaced by a $\mathbb{N}^N$-graduation, which means that we replace the single coupling constant by N coupling constants.

We look for $\mathbb{N}^N$-graduations of the bialgebra of Feynman graphs $\mathcal{H}_{\mathcal{FG}(\mathcal{T})}$ combinatorially defined using:

- the number of vertices of each type.
- The number of internal edges of each type.
- The external structure.

For any Feynman graph G, we define vectors $V_G \in \mathbb{N}^k$ and $S_G \in \mathbb{N}^{k+l}$:

$$(V_G)_i = \sharp\{\text{vertices of } G \text{ of type } i\},$$
$$(S_G)_j = \sharp\{\text{connected components of } G \text{ of type } j\}.$$

Proposition

Such a graduation is given by a matrix $C \in M_{N,k}(\mathbb{Q})$ such that for any Feynman graph G:

$$deg(G) = CV_G - (C\ 0)S_G.$$

The incidence matrix of the theory $\mathcal{T}$ is:

$$A_{\mathcal{T}} = (a_{e,v})_{e \text{ half edge of } \mathcal{T}, v \text{ vertex type of } \mathcal{T}},$$

where $a_{e,v}$ is the multiplicity of e in the multiset v.

$$A_{QED} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad A_{QCD} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 4 \end{pmatrix} \quad A_{\varphi^n} = (n).$$

For the loop number:

$$C = \frac{(1 \dots 1) A_{\mathcal{T}}}{4} - (1 \dots 1).$$

Fixing such a matrix C, we now consider the system given by:

Dyson-Schwinger system $(S_{\mathcal{T}})$ associated to $\mathcal{T}$

if $1 \leq i \leq k+l$:

$$X_i = \sum_{G \in P_i} \prod_{i=1}^{N} \alpha_i^{deg_i(G)} B_G \left(\prod_{j=1}^{k} (1+X_j)^{v_i(G)} \prod_{j=k+1}^{k+l} (1-X_j)^{-e_j(G)} \right).$$

We put:

$$X_i = \sum_{a \in \mathbb{N}^N} \prod_{i=1}^{N} \alpha_i^{a_i} X_i(a).$$

Is the subalgebra $\mathcal{H}_{(S)}$ generated by the $X_i(a)$ a subbialgebra?

If $a, b \in \mathbb{K}$, we denote by $F_{a,b}(X)$ the formal series:

$$F_{a,b}(X) = \sum_{k=0}^{\infty} \frac{a(a-b)\ldots(a-b(k-1))}{k!} X^k = \begin{cases} (1+bX)^{\frac{a}{b}} \text{ if } b \neq 0, \\ e^{aX} \text{ if } b = 0. \end{cases}$$

Let $D_{M,N} = [M] \times \mathbb{N}_*^N$. If $(i, a) \in D_{M,N}$, $deg(i, a) = a \in \mathbb{N}_*^N$. We fix:

1. Let $[M] = I_0 \sqcup \ldots \sqcup I_k$ be a partition of $[M]$, such that $I_1, \ldots, I_k \neq \emptyset$.
2. $A_1, \ldots, A_k \in \mathbb{K}^N$, $b_1, \ldots, b_p \in \mathbb{K}$, and $b_p^{(i)} \in \mathbb{K}$ for all $i \in I_0$ and $p \in [k]$.

Theorem

We consider the system: $\forall 1 \leq p \leq M$, $\forall i \in I_p$, $\forall i' \in I_0$:

$$X_i = \sum_{a \in \mathbb{N}_*^N} \alpha^a B_{(i,a)} \left(\prod_{q=1}^{k} F_{A_q \cdot a, b_q} \left(\sum_{j \in I_q} X_j \right) \left(1 + b_p \sum_{j \in I_p} X_j \right) \right),$$

$$X_{i'} = \sum_{a \in \mathbb{N}_*^N} \alpha^a B_{(i',a)} \left(\prod_{q=1}^{k} F_{A_q \cdot a, b_q} \left(\sum_{j \in I_q} X_j \right) \prod_{q=1}^{k} F_{b_q^{(i')}, b_q} \left(\sum_{j \in I_q} X_j \right) \right).$$

The subalgebra generated by the components of the solution of this system is a subbialgebra.

Idea of the proof:

1. Introduce a family of prelie algebras.
2. Classify them.
3. See them as a quotient of free prelie algebras (Chapoton-Livernet description).
4. Using the Oudom-Guin construction, see their enveloping algebras as a quotients of Grossman-Larson algebras.
5. Dually, see the dual of their enveloping algebras as subalgerbas of Connes-kreimer Hopf algebras.

We now consider a theory of Feynman graphs $\mathcal{T}$ with k vertex types and l edges; $M = k + l$.

- We give $\mathcal{H}_{\mathcal{FG}(\mathcal{T})}$ a $\mathbb{N}^N$-graduation induced by a matrix $C \in M_{N,k}(\mathbb{Q})$.
- We consider the subalgebra $\mathcal{H}_{(S)}$ generated by the components of the solution of the system associated to $\mathcal{T}$: if $1 \leq i \leq k + l$,

$$X_i = \sum_{G \in P_i} \alpha^{deg(G)} B_G \left(\prod_{j=1}^{k} (1 + X_j)^{v_i(G)} \prod_{j=k+1}^{k+l} (1 - X_j)^{-e_j(G)} \right).$$

Theorem

If $rank(C) = k$, then (S) is a system of the preceding form, with parameters:

$$(A_1 \ldots A_k) = \begin{pmatrix} I_k & 0 \\ A'' & 0 \end{pmatrix} \qquad\qquad b_i = 0$$

If $C \in GL_k(\mathbb{Q})$, $A'' = -A'_T \in M_{l,k}(\mathbb{Q})$, with:

$$(a'_T)_{i,j} = \begin{cases} \frac{a_{e,j}}{2} \text{ if the } i\text{-th edge is } \{e, e\}, \\ \frac{a_{e,j} + a_{e',j}}{2} \text{ if the } i\text{-th edge is } \{e, e'\},\ e \neq e'. \end{cases}$$

$$A'_{QED} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \quad A'_{QCD} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 2 \end{pmatrix} \quad A'_{\varphi^n} = \left(\frac{n}{2}\right).$$

Question

What is the minimal rank m of C such that $\mathcal{H}_{(S)}$ is a subbialgebra?

We proved that $m \leq k$, the number of vertex types of $\mathcal{T}$.
For *QED* and φ^n, $m = k = 1$.

Proposition

For QCD, $m = k = 4$.

Idea of the proof: produce enough primitive QCD Feynman graphs.

In QCD, we take:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 2 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}.$$

If G is a QCD Feynman graph, then:

$deg(G) = \Big(deg$  $(G),\ deg$  $(G),\ deg$  $(G),\ \ell(G) \Big),$

where $deg_e(G)$ is the number of internal and external edges of type e.

We fix a matrix $B \in M_{p,q}(\mathbb{K})$. For all $1 \leq i \leq p$:

$$\mathbf{G}_i = \{x_i(1+F) \mid F \in \mathbb{K}[[x_1,\ldots,x_p,y_1,\ldots,y_q]]_+\} \subseteq \mathbb{K}[[x_1,\ldots,x_p,y_1,\ldots,y_q]]_+.$$

Faà di Bruno group

Let $\mathbf{G}_B = \mathbf{G}_1 \times \ldots \times \mathbf{G}_p \subseteq \mathbb{K}[[x_1,\ldots,x_p,y_1,\ldots,y_q]]^p$, with the product defined in the following way: if $F = (F_1,\ldots,F_p)$ and $G = (G_1,\ldots,G_p) \in \mathbf{G}_B$,

$$F \bullet G = G\left(\begin{array}{c} F_1,\ldots,F_p, \\ y_1\left(\frac{F_1}{x_1}\right)^{B_{1,1}}\cdots\left(\frac{F_p}{x_p}\right)^{B_{1,p}},\ldots,y_q\left(\frac{F_1}{x_1}\right)^{B_{q,1}}\cdots\left(\frac{F_p}{x_p}\right)^{B_{q,p}} \end{array}\right).$$

Module over G_B

Let V_0 be the group $(\mathbb{K}[[x_1, \ldots, x_p, y_1, \ldots, y_q]]_+, +)$. The group $\mathbf{G}_B$ acts by automorphisms on V_0 by:

$$\forall F \in \mathbf{G}_B,\ \forall P \in V_0,\ F \hookrightarrow P = P\left(F, y\left(\frac{F}{x}\right)^B\right).$$

Group associated to a theory of Feynman graphs

If $rank(C) = k$, the bialgebra $\mathcal{H}_{(S)}$ is isomorphic to the coordinate algebra of the group:

$$V_0^l \rtimes G_{A''}.$$

Thank you