Renormalization for Stochastic PDEs with Non-Gaussian Noises

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Stochastic PDEs

- Kardar-Parisi-Zhang

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi$$

- Dynamical Φ^4

$$\partial_t \phi = \Delta \phi - \phi^3 + \xi$$

- SPDE with multiplicative noise (has Itô solution)

$$\partial_t u = \partial_x^2 u + H(u) + G(u)\xi$$

Stochastic PDEs: renormalization

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$$\partial_t \phi = \Delta \phi - \phi^3 + \xi$$

$$\partial_t u = \partial_x^2 u + H(u) + G(u)\xi$$

To define solutions:
1) Replace ξ by ξ_ε, a sequence of smooth Gaussian fields.
2) As $\varepsilon \to 0$, $\xi_\varepsilon \to \xi$. However, the smooth solutions do not converge. - Therefore needs renormalization (add counter-terms).
3) Take limit $\varepsilon \to 0$ with counter-terms.

Stochastic PDEs: renormalization

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi$$

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Q: How about Non-Gaussian approximation $\zeta_\varepsilon \to \xi$?

1) Assuming ζ_ε is mixing, $\zeta_\varepsilon \to \xi$ as $\varepsilon \to 0$ by standard CLT.

2) However, the smooth solutions (for noises ζ_ε) do not converge, even with the above renormalization!

3) Needs extra renormalization (counter-terms) due to non-Gaussianity.

More concrete assumptions on the noise

We consider the following general class of Non-Gaussian noises.

- ζ_ε is rescaled field of ζ:

$$\zeta_\varepsilon = \varepsilon^{-D/2}\zeta(x/\varepsilon, t/\varepsilon^2)$$

  where ζ satisfies:

  - Mixing: $\zeta(z)$ and $\zeta(z')$ are independent whenever $|z - z'| > 1$. (Or, dependence decays exponentially on scale O(1).)
  - Bounded moments.
  - Continuity / Smoothness.
  - Stationary and centered.

Result 1: (non-standard) CLT for KPZ

(Hairer & S. 2015)

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + (\partial_x h_\varepsilon)^2 + \zeta_\varepsilon$$

where ζ_ε is non-Gaussian. Then

$$h_\varepsilon(x - v_{hor} t, t) - v_{ver}^{(\varepsilon)} t \to h$$

h is the same solution to KPZ with (Gaussian) white noise; the speeds v_{hor}, $v_{ver}^{(\varepsilon)}$ depend on the first four cumulants of ζ explicitly.

- (Hairer & Quastel 2015, Gubinelli & Perkowski 2016): polynomial nonlinearities in $\partial_x h$ - universality result.
- General continuous growth models

  $$\partial_t h = \text{smoothing} + \text{lateral growth (i.e.interaction)} + \text{randomness}$$

  should scale to KPZ (under "weak asymmetry assumption").

Result 2: Universality of Phi4_3

(S. & Xu 2016) A general class of phase coexistence models:

$$\partial_t u = \Delta u + \varepsilon V'(u) + \zeta$$

Rescale $u_\varepsilon(x,t) = \varepsilon^{-\frac{1}{2}} u(\varepsilon^{-1}x, \varepsilon^{-2}t)$. Under "pitchfork" assumption, u_ε converges to solution of

$$\partial_t u = \Delta u - u^3 + \xi$$

- (Hairer & Xu 2016) proved universality for gaussian case.

Result 3: Wong-Zakai theorem

- (A result in stochastic analysis)

$$dX_t = H(X_t)dt + G(X_t)dB$$

$B_\varepsilon \to B$, has to subtract $\frac{1}{2}G'(X_\varepsilon)G(X_\varepsilon)$ to obtain Itô limit.

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- (Hairer & Pardoux 2014) Approximating Itô solution to

$$\partial_t u = \partial_x^2 u + H(u) + G(u)\xi$$

Let ξ_ε be smooth Gaussian, $\xi_\varepsilon \to \xi$.

$$\begin{aligned}\partial_t u_\varepsilon =& \partial_x^2 u_\varepsilon + H(u_\varepsilon) - \varepsilon^{-1} c_0 G'(u_\varepsilon)G(u_\varepsilon) \\ & - c_1 G'(u_\varepsilon)^3 G(u_\varepsilon) - c_2 G''(u_\varepsilon)G'(u_\varepsilon)G(u_\varepsilon)^2 + G(u_\varepsilon)\xi_\varepsilon\end{aligned}$$

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- (Chandra & S. in progress) Let ζ_ε be smooth non-Gaussian

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + H(u_\varepsilon) - H_1(u_\varepsilon) - H_2(u_\varepsilon) + G(u_\varepsilon)\zeta_\varepsilon$$

$$H_1(u_\varepsilon) = -\varepsilon^{-1} c_0 G'(u_\varepsilon)G(u_\varepsilon) - \varepsilon^{-\frac{1}{2}} c^{(1)} G'(u_\varepsilon)^2 G(u_\varepsilon) - \varepsilon^{-\frac{1}{2}} c^{(2)} G''(u_\varepsilon)G(u_\varepsilon)^2$$

$$H_2(u) = -c^{(\alpha)} G'''(u)G(u)^3 - c^{(\beta)} G'(u)^3 G(u) - c^{(\gamma)} G''(u)G'(u)G(u)^2$$

Perturbative solutions

$$\partial_t \phi = \Delta\phi - \lambda\phi^3 + \xi$$

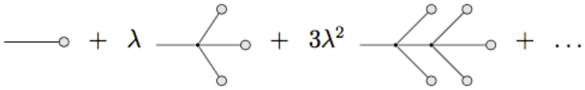
Let $\phi = \phi_0 + \lambda\phi_1 + \lambda^2\phi_2 \dots$ Then

$$\partial_t \phi_0 = \Delta\phi_0 + \xi$$

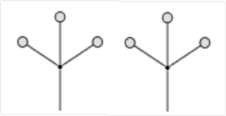
$$\partial_t \phi_1 = \Delta\phi_1 - \phi_0^3$$

$$\dots\dots$$

Let $P = (\partial_t - \Delta)^{-1}$. Solve them: $\phi_0 = P * \xi$, $\phi_1 = -P * (\phi_0^3)$, etc.



Correlation (Gaussian noise)

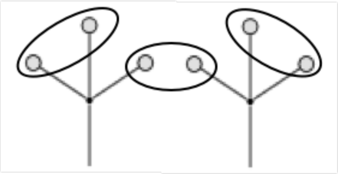


each copy $= \int (\prod \text{heat kernels})\zeta(x_1)\zeta(x_2)\zeta(x_3)$

Wick theorem:

$$\mathbf{E}(\prod_{i\in A}\zeta(x_i)) = \sum_{\text{pairings}\,\pi}\prod_{(i,j)\in\pi}\mathbf{E}(\zeta(x_i)\zeta(x_j))$$

One of the pairings:



Correlation (Gaussian noise)

Wiener chaos decomposition ($X^3 = (X^3 - 3X) + 3X$)

$$\begin{aligned}\zeta(x_1)\zeta(x_2)\zeta(x_3) &= :\zeta(x_1)\zeta(x_2)\zeta(x_3): \\ &+ \mathbf{E}(\zeta(x_1)\zeta(x_2)) :\zeta(x_3): + \mathbf{E}(\zeta(x_1)\zeta(x_3)) :\zeta(x_2): + \mathbf{E}(\zeta(x_2)\zeta(x_3)) :\zeta(x_1):\end{aligned}$$

In general,

$$\prod_{i=1}^{n} \zeta(x_i) = \sum_A \mathbf{E}\Big(\prod_{i \in A} \zeta(x_i)\Big) :\prod_{j \notin A} \zeta(x_j):$$

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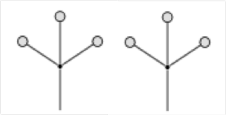
In general,

$$\prod_{i=1}^{n} \zeta(x_i) = \sum_{A} \mathbf{E}\Big(\prod_{i \in A} \zeta(x_i)\Big) :\prod_{j \notin A} \zeta(x_j):$$

$$= \sum_{A} \Big(\sum_{\text{pairings of } A} \prod \mathbf{E}(\zeta\zeta) \Big) :\prod_{j \notin A} \zeta(x_j):$$

- Wick renormalization: kill the divergent chaoses.

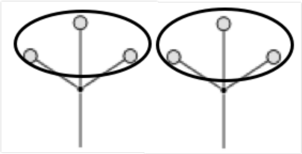
Correlation (Non-Gaussian)



each copy $= \int (\prod \text{heat kernels})\zeta(x_1)\zeta(x_2)\zeta(x_3)$

Generalized Wick theorem:

$$\mathbf{E}(\prod_{i\in A}\zeta(x_i)) = \sum_{\text{partitions}_\pi} \prod_{B\in\pi} \mathbf{E}_c\{\zeta(x_i)|i\in B\}$$



Correlation (Non-Gaussian)

A generalized Wiener chaos decomposition

$$\zeta(x_1)\zeta(x_2)\zeta(x_3) = :\zeta(x_1)\zeta(x_2)\zeta(x_3): + \mathbf{E}_c(\zeta(x_1),\zeta(x_2)) :\zeta(x_3): + \mathbf{E}_c(\zeta(x_1),\zeta(x_3)) :\zeta(x_2): + \mathbf{E}_c(\zeta(x_2),\zeta(x_3)) :\zeta(x_1): + \mathbf{E}_c(\zeta(x_1),\zeta(x_2),\zeta(x_3))$$

In general,

$$\prod_{i=1}^{n}\zeta(x_i) = \sum_A \mathbf{E}\Big(\prod_{i\in A}\zeta(x_i)\Big) :\prod_{j\notin A}\zeta(x_j): = \sum_A \Big(\sum_{\text{partitions of } A}\prod \text{cumulants}\Big) :\prod_{j\notin A}\zeta(x_j):$$

- Further renormalization: May need to kill divergent graphs with higher cumulants.

Technical difficulties

After renormalization, to prove the remaining graphs are well-bounded,

- Do not have "Hyper-contractivity" or "Equivalence of moments" as in Gaussian case, which bounds higher moments by second moment automatically.
- Do not have martingale structure, therefore no "Burkholder-Davis-Gundy" inequaltiy which essentially reduces higher moments to second moment.
- Therefore, we have to bound moments of arbitrary orders by hand.

Power counting criteria

Given a graph H, every edge e represents a kernel with degree of singularity a_e.
For every subgraph $\bar{H} \subset H$

$$\sum_{e \in \mathcal{E}(\bar{H})} a_e < D \left(|\bar{H}_{in}| + \frac{1}{2}(|\bar{H}_{ex}| - 1 - \mathbf{1}_{\bar{H}_{ex}=\emptyset}) \right)$$

where D is space-time dimension.

- Actually four conditions.
- Hairer-Quastel, Hairer-S., Chandra-S., Chandra-Hairer

Result 1: (non-standard) CLT for KPZ

(Hairer & S. 2015)

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + (\partial_x h_\varepsilon)^2 + \zeta_\varepsilon$$

where ζ_ε is non-Gaussian. Then

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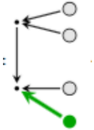
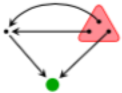


contains a 0-th order chaos



$\approx \varepsilon^{-\frac{1}{2}}$

Result 2: Universality of Phi4_3 with non-Gaussian noise

(S. & Xu)

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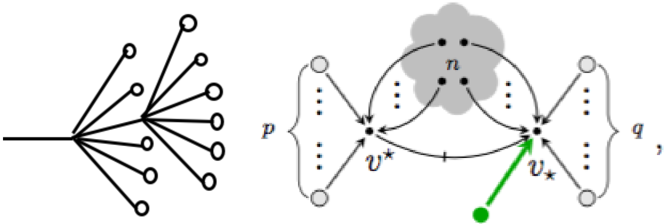

Result 3: Wong-Zakai theorem with non-Gaussian noise

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