On the renormalization problem of multiple zeta values

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Outline

1. Multiple zeta values
   - Introduction
   - Linear relations
2. Extensions of multiple zeta values
   - Examples in the literature
   - General construction principle
3. Renormalization group
   - General case
   - Multiple zeta value case

Multiple zeta values

Multiple zeta values (MZVs) are given by the following nested series:

$$\zeta(k_1,\ldots,k_n) := \sum_{m_1>\cdots>m_n>0} \frac{1}{m_1^{k_1}\cdots m_n^{k_n}}.$$

- Usually the MZVs are studied for positive integers $k_1,\ldots,k_n \in \mathbb{N}$ with $k_1 \geq 2$.
- The series is convergent for $k_1,\ldots,k_n \in \mathbb{Z}$ with

$$k_1+\cdots+k_j > j \qquad \text{for } j=1,\ldots,n.$$

- The integer n is called the *depth* and the sum $k_1+\cdots+k_n$ is the *weight*.

For $n = 1$ we obtain the well-known *Riemann zeta function*

$$\zeta_1(s) := \sum_{m>0} \frac{1}{m^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}},$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

Facts

- ζ_1 has an analytic continuation to $\mathbb{C} \setminus \{1\}$ and a pole in $s = 1$.
- Functional equation: $\zeta_1(1-s) = \frac{1}{(2\pi)^s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta_1(s)$.
- $\zeta_1(2k) = -\frac{(2\pi i)^{2k} B_{2k}}{2(2k)!}$ for $k \in \mathbb{N}$.
- $\zeta_1(-k) = -\frac{B_{k+1}}{k+1}$ for $k \in \mathbb{N}_0$.

Renormalization problem of MZVs (rough version)

Provide a systematic extension procedure for MZVs to arbitrary integer arguments...

The *multiple zeta function* ζ_n is also defined by the nested series

$$\zeta_n(s_1, \dots, s_n) := \sum_{m_1 > \cdots > m_n > 0} \frac{1}{m_1^{s_1} \cdots m_n^{s_n}}.$$

Theorem (Akiyama, Egami, Tanigawa 2001)

The function $\zeta_n(s_1, \dots, s_n)$ admits a meromorphic extension to $\mathbb{C}^n$. The subvariety $\mathcal{S}_n \subseteq \mathbb{C}^n$ of singularities is given by

$$\begin{aligned} s_1 &= 1 \text{ or} \\ s_1 + s_2 &= 2, 1, 0, -2, -4, \dots \text{ or} \\ s_1 + \cdots + s_j &\in \mathbb{Z}_{\leq j} \quad (j = 3, 4, \dots, n). \end{aligned}$$

- In the Riemann zeta case ($n = 1$) we have $\mathbb{Z}_{\leq 0} \cap \mathcal{S}_1 = \emptyset$.
- For $n \geq 3$ we observe $(\mathbb{Z}_{\leq 0})^n \subseteq \mathcal{S}_n$.

Renormalization problem of MZVs (refined version)

Provide a systematic extension procedure for MZVs to arbitrary integer arguments such that

(A) the meromorphic continuation is verified whenever it is defined...

MZVs exhibit more structure:
The $\mathbb{Q}$-vector space spanned by the MZVs is denoted by

$$\mathcal{M} := \left\langle \zeta(\mathbf{k}) \colon \mathbf{k} \in \mathbb{N}^n, k_1 \geq 2, n \in \mathbb{N} \right\rangle_{\mathbb{Q}}.$$

The vector space $\mathcal{M}$ is an algebra with two different products:

- quasi-shuffle product,
- shuffle product.

Quasi-shuffle product

Using the defining series representation of MZVs one can show that the product of two MZVs is again a $\mathbb{Q}$-linear combination of MZVs.

Example: Nielsen's reflexion formula

For integers $a, b \geq 2$ we have

$$\begin{aligned}\zeta(a)\zeta(b) &= \sum_{m>0} \frac{1}{m^a} \sum_{n>0} \frac{1}{n^b} \\ &= \sum_{m>n>0} \frac{1}{m^a n^b} + \sum_{n>m>0} \frac{1}{n^b m^a} + \sum_{m>0} \frac{1}{m^{a+b}} \\ &= \zeta(a,b) + \zeta(b,a) + \zeta(a+b).\end{aligned}$$

- This can be generalized to arbitrary depth.
- We call these relations *quasi-shuffle relations*.

Shuffle product

Multiple zeta values are periods, i.e., we have the following integral formula:

$$\zeta(k_1,\dots,k_n) = \int\limits_{1>t_1>\dots>t_k>0} \omega_1(t_1)\cdots\omega_k(t_k),$$

where $k := k_1 + \dots + k_n$ and $\omega_i(t) := \frac{dt}{1-t}$ if $i \in \{k_1, k_1+k_2, \dots, k_1+\dots+k_n\}$ and $\omega_i(t) := \frac{dt}{t}$ otherwise.

Example

$$\begin{aligned}\zeta(2)^2 &= \int_{\substack{1>t_1>t_2>0\\ 1>\tilde{t}_1>\tilde{t}_2>0}} \frac{dt_1}{t_1}\frac{dt_2}{1-t_2}\frac{d\tilde{t}_1}{\tilde{t}_1}\frac{d\tilde{t}_2}{1-\tilde{t}_2}\\ &= 4\zeta(3,1) + 2\zeta(2,2).\end{aligned}$$

- This can be generalized to arbitrary depth and weight.
- We call these relations *shuffle relations*.

Regularized double shuffle relations

- Combining quasi-shuffle and shuffle relations we obtain the so-called *double shuffle relations*. For example

$$4\zeta(3,1)+2\zeta(2,2) \overset{\text{shuffle}}{=} \zeta(2)\zeta(2) \overset{\text{qu.-shuffle}}{=} 2\zeta(2,2)+\zeta(4)$$

  leads to $\zeta(4)=4\zeta(3,1)$.
- One can extend the quasi-shuffle and shuffle relations by regularization relations. For this one identifies the divergent zeta value "$\zeta(1)$" with a formal variable T. They are called *regularized double shuffle relations*. For example $\zeta(3)=\zeta(2,1)$.

Conjecture

All linear relations among MZVs are induced by regularized double shuffle relations.

Renormalization problem of MZVs (final version)

Provide an extension procedure for MZVs to arbitrary integer arguments such that

(A) the meromorphic continuation is verified whenever it is defined and

(B) the quasi-shuffle relation is satisfied.

Naturally some questions arise:

- Are there solutions to the previous problem? (existence, uniqueness)
- What are the relations between different solutions?

Word algebraic description of the quasi-shuffle relations

- Introduce the infinite alphabet $Y := \{z_k : k \in \mathbb{Z}\}$.
- Y^* denotes the set of words with letters in Y.
- $\mathcal{H} := \mathbb{Q}\langle Y\rangle$ be the free (non-commutative) algebra generated by Y.
- We define the *quasi-shuffle product* $*: \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ by
  (i) $\mathbf{1} * w := w * \mathbf{1} := w$,
  (ii) $z_m u * z_n v := z_m(u * z_n v) + z_n(z_m u * v) + z_{m+n}(u * v)$,

  for words $w, u, v \in \mathcal{H}$ and $m, n \in \mathbb{Z}$.
- The unit map is given by $u: \mathbb{Q} \to \mathcal{H}$, $u(\lambda) = \lambda\mathbf{1}$.

Example

For $a, b, c \in \mathbb{Z}$ we have

$$z_a * z_b z_c = z_a z_b z_c + z_b z_a z_c + z_b z_c z_a + z_b z_{a+c} + z_{a+b} z_c,$$

which reflects the multiplication of the (formal) series

$$\zeta(a)\zeta(b,c) = \zeta(a,b,c) + \zeta(b,a,c) + \zeta(b,c,a) + \zeta(b,a+c) + \zeta(a+b,c).$$

- Let $Y_+ := \{z_k \colon k \in \mathbb{N}\}$ and $w := z_{k_1} \cdots z_{k_n} \in W := Y_+^* \setminus z_1 Y_+^*$. We define

$$\zeta^*(w) := \zeta(k_1, \ldots, k_n).$$

Lemma (Hoffman 1997)

The map ζ^* is an algebra morphism, i.e., $\zeta^*(w_1)\zeta^*(w_1) = \zeta^*(w_1 * w_2)$ for $w_1, w_2 \in W$.

- The coproduct $\Delta \colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is given by deconcatenation

$$\Delta(w) := \sum_{uv=w} u \otimes v$$

for any word $w \in \mathcal{H}$.

- The counit $\varepsilon : \mathcal{H} \to \mathbb{Q}$ is defined by $\varepsilon(\mathbf{1}) = 1$, and $\varepsilon(w) = 0$ for $w \neq \mathbf{1}$.

- The reduced coproduct $\widetilde{\Delta}$ is given by

$$\widetilde{\Delta}(w) := \Delta(w) - w \otimes \mathbf{1} - \mathbf{1} \otimes w$$

for $w \in \ker(\varepsilon)$. In Sweedler's notation we have

$$\widetilde{\Delta}(w) = \sum_{(w)} w' \otimes w''.$$

- $(\mathcal{H}, *, \Delta)$ is a filtered and connected bialgebra. Hence, it is a Hopf algebra with antipode $S : \mathcal{H} \to \mathcal{H}$ given by $S(\mathbf{1}) = \mathbf{1}$ and

$$S(w) = -w - \sum_{(w)} S(w') * w'' = -w - \sum_{(w)} w' * S(w'')$$

for any word $w \in \ker(\varepsilon)$.

- For linear maps ϕ, ψ from $\mathcal{H}$ to an algebra $\mathcal{A}$ the *convolution product* is defined by

$$\phi \star \psi := m_{\mathcal{A}} \circ (\phi \otimes \psi) \circ \Delta.$$

Examples in the literature

- Guo/Zhang (2008): Solution for *non-positive* arguments, e.g.,

$$\zeta_{\mathsf{GZ}}(-1,-3)=\frac{83}{64512}.$$

- Manchon/Paycha (2010): Solution for *arbitrary* arguments, e.g.,

$$\zeta_{\mathsf{MP}}(-1,-3)=\frac{1}{840}.$$

- Ebrahimi-Fard/Manchon/Singer (2015): One-parameter family of solutions for *strictly negative* arguments, e.g.,

$$\zeta_{\mathsf{EMS},t}(-1,-3)=\frac{1}{8064}\frac{166t^2+166t+31}{(4t+3)(4t+1)}$$

for $t\in\{s\in\mathbb{C}\colon\ \mathrm{Re}(s)>0\}$.

General construction principle

Theorem (Connes, Kreimer 2000, Manchon 2008)

Let $\mathcal{H}$ be a graded or filtered Hopf algebra over a ground field k, and let $\mathcal{A}$ a commutative unital k-algebra equipped with a renormalization scheme $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ and the corresponding idempotent Rota–Baxter operator π, where $\mathcal{A}_- = \pi(\mathcal{A})$ and $\mathcal{A}_+ = (\mathrm{Id} - \pi)(\mathcal{A})$. The character $\psi\colon \mathcal{H} \to \mathcal{A}$ admits a unique Birkhoff decomposition

$$\psi_- \star \psi = \psi_+,$$

where $\psi_- : \mathcal{H} \to k\mathbf{1} \oplus \mathcal{A}_-$ and $\psi_+ : \mathcal{H} \to \mathcal{A}_+$ are characters.

- Quasi-shuffle Hopf algebra $(\mathcal{H}, *, \Delta)$.
- Renormalization scheme: *Minimal subtraction*, i.e., let $\mathcal{A} := \mathbb{C}[\lambda^{-1}, \lambda]]$, $\mathcal{A}_- := \lambda^{-1}\mathbb{C}[\lambda^{-1}]$ and $\mathcal{A}_+ := \mathbb{C}[[\lambda]]$ with $\pi : \mathcal{A} \to \mathcal{A}_-$ by

$$\pi\left(\sum_{n=-l}^{\infty} a_n \lambda^n\right) := \sum_{n=-l}^{-1} a_n \lambda^n.$$

- *Regularization process*: The defining series of MZVs is replaced by

$$\zeta^{(\lambda)}(z_{k_1}\cdots z_{k_n}) := \sum_{m_1>\cdots>m_n>0} f_{\lambda,k_1}(m_1)\cdots f_{\lambda,k_n}(m_n) \in \mathcal{A}.$$

  The deformations $f_{\lambda,\ell}(x)$ are defined by
  - GZ: $z_\ell \mapsto f_{\lambda,\ell}(x) := \frac{\exp(-\ell x\lambda)}{x^\ell}$,
  - MP: $z_\ell \mapsto f_{\lambda,\ell}(x) := \frac{1}{x^{\ell-\lambda}}$,
  - EMS: $z_\ell \mapsto f_{\lambda,\ell}(x) := \frac{q^{|\ell|x}}{(1-q^x)^\ell}$, $\lambda = \log(q)$.
- The renormalized MZVs are then obtained by $\lim_{\lambda\to 0}\zeta_+^{(\lambda)}$.

Renormalization group

- Let $\mathcal{H}$ be a connected, filtered Hopf algebra over a field k of characteristic zero.
- Let $\mathcal{A}$ be a commutative unital k-algebra.
- The set $G_{\mathcal{A}}$ of unital algebra morphisms from $\mathcal{H}$ to $\mathcal{A}$ is a group w.r.t. to the convolution product $\star$.

Lemma

Let $N \subseteq \mathcal{H}$ be a left coideal with respect to the reduced coproduct, i.e., $\widetilde{\Delta}(N) \subseteq N \otimes \mathcal{H}$ and $\varepsilon(N) = \{0\}$. The set

$$T_{\mathcal{A}} := \left\{\phi \in G_{\mathcal{A}} \colon \phi|_N = 0\right\}$$

is a subgroup of $(G_{\mathcal{A}}, \star, e)$.

Sketch of proof.

Obviously, $e \in T_{\mathcal{A}}$. Let $\phi, \psi \in T_{\mathcal{A}}$. Since $G_{\mathcal{A}}$ is a group $\phi \star \psi^{-1} \in G_{\mathcal{A}}$. Further, for any $w \in N$, we have

$$\begin{aligned}(\phi \star \psi^{-1})(w) &= \big(\phi \star (\psi \circ S)\big)(w) \\ &= \phi(w) + \psi\big(S(w)\big) + \sum_{(w)} \phi(w')\psi(S(w'')) \\ &= \phi(w) - \psi(w) - \sum_{(w)} \psi(w')\psi(S(w'')) \\ &\quad + \sum_{(w)} \phi(w')\psi(S(w'')) = 0\end{aligned}$$

using the fact that by definition N is a left-coideal of $\widetilde{\Delta}$. $\square$

Definition

The group $T_{\mathcal{A}}$ is called the *renormalization group* associated to N.

Let $\zeta : N \to \mathcal{A}$ be a partially defined character on $\mathcal{H}$, i.e., a linear map such that $\zeta(v.w) = \zeta(v)\zeta(w)$ as long as v, w and the product $v.w$ belong to N. We define the set of all possible renormalizations with target algebra $\mathcal{A}$ by

$$X_{\mathcal{A},\zeta} := \{\alpha \in G_{\mathcal{A}} : \alpha|_N = \zeta\}.$$

Theorem

The set $X_{\mathcal{A},\zeta}$ is a principal homogenous space for the group $T_{\mathcal{A}}$. More precisely, the left group action

$$\begin{aligned} T_{\mathcal{A}} \times X_{\mathcal{A},\zeta} &\longrightarrow X_{\mathcal{A},\zeta} \\ (\phi, \alpha) &\longmapsto \phi \star \alpha \end{aligned}$$

is free and transitive.

Sketch of proof.

The group action is well-defined, since for $\phi \in T_{\mathcal{A}}$, $\alpha \in X_{\mathcal{A},\zeta}$ and $w \in N$

$$(\phi \star \alpha)(w) = \phi(w) + \alpha(w) + \sum_{(w)} \phi(w')\alpha(w'') = \alpha(w) = \zeta(w),$$

using that $\phi|_N = 0$. Freeness is obvious. For transitivity let $\alpha, \beta \in X_{\mathcal{A},\zeta}$. Then for $w \in N$

$$\begin{aligned}(\alpha \star \beta^{-1})(w) &= \big(\alpha \star (\beta \circ S)\big)(w) = \alpha(w) + \beta\big(S(w)\big) + \sum_{(w)} \alpha(w')\beta(S(w'')) \\ &= \alpha(w) - \beta(w) - \sum_{(w)} \beta(w')\beta(S(w'')) + \sum_{(w)} \alpha(w')\beta(S(w'')) \\ &= (\alpha - \beta)(w) + \sum_{(w)} (\alpha - \beta)(w')\beta(S(w'')) = 0,\end{aligned}$$

using $\alpha|_N = \beta|_N = \zeta$. $\square$

Renormalization group for MZVs

We apply the general framework to the following data:

- Quasi-shuffle Hopf algebra $\mathcal{H} := (\mathbb{Q}\langle Y\rangle, *, \Delta)$, $Y := \{z_k : k \in \mathbb{Z}\}$
- Target space $\mathcal{A} := \mathbb{C}$.
- Left coideal N:

Definition

A word $w := z_{k_1} \cdots z_{k_n} \in Y^*$ is called *non-singular* if and only if

- $k_1 \neq 1$ and
- $k_1 + k_2 \notin \{2, 1, 0, -2, -4, \ldots\}$ and
- $k_1 + \cdots + k_j \notin \mathbb{Z}_{\leq j}$ for $j \geq 3$.

The linear span of all non-singular words is denoted by N

Lemma

The space N is a left coideal for the reduced coproduct $\widetilde{\Delta}$ and invariant under contractions, e.g., $z_{k_1} z_{k_2} z_{k_3} z_{k_4} z_{k_5} \mapsto z_{k_1+k_2} z_{k_3} z_{k_4+k_5}$.

- For words $w = z_{k_1} \cdots z_{k_n} \in N$ the partially defined character $\zeta^*\colon N \to \mathbb{C}$ is given by

$$\zeta^*(w) := \zeta_n(k_1, \dots, k_n),$$

which is either convergent or can be defined by analytic continuation.

Renormalization problem of MZVs

- Manchon/Paycha: There exists a solution $\phi \in X_{\mathbb{C},\zeta^*}$.
- Main theorem: All solutions lie on a single orbit, i.e., $X_{\mathbb{C},\zeta^*} = T_{\mathbb{C}} \star \phi$.
- By restricting the Hopf algebra to words generated by $Y_{<0} := \{z_k \colon k \in \mathbb{Z}_{<0}\}$ and $Y_{\leq 0} := \{z_k \colon k \in \mathbb{Z}_{\leq 0}\}$ we can compare all other values found in the literature with each other.

Question

How large is the renormalization group $T_{\mathbb{C}}$?

Let $\mathcal{A}$ be a commutative unital algebra.

Facts

- The two-sided ideal $\mathcal{N}$ generated by N is a Hopf ideal of $\mathcal{H}$.
- The renormalization group $T_{\mathcal{A}}$ is isomorphic to the group of characters of the Hopf algebra $\mathcal{H}/\mathcal{N}$.

Theorem

The renormalization group $T_{\mathcal{A}}$ is pro-unipotent and can be identified with the space $\mathcal{L}(W, \mathcal{A})$ of linear maps from W to $\mathcal{A}$, where $W = \pi_1(\mathcal{H}/\mathcal{N})$ and $\pi_1 := \log^\star \mathrm{Id}_{\mathcal{H}/\mathcal{N}}$ is the Eulerian idempotent.

Theorem

The Lie algebra $\mathfrak{t}_{\mathcal{A}}$ of the pro-unipotent renormalization group $T_{\mathcal{A}}$ is infinite-dimensional.

Summary

- We have shown that the renormalization problem of MZVs has infinitely many solutions.
- The relation between two solutions is described by a transfer character which is an element of the renormalization group.
- The group action is independent of a specific renormalization procedure and comprises all possible solutions of the renormalization problem.

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Thank you for your attention!