q-Multiple zeta values II: from regularization to shuffle renormalization

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Outline

Renormalization problem revisited

On Monday we regarded *multiple zeta values* (MZVs) given by

$$\zeta(k_1, \ldots, k_n) := \sum_{m_1 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}$$

for integers $k_1 \geq 2,\ k_2, \ldots, k_n \geq 1$ and discussed the following problem:

Renormalization problem of MZVs (quasi-shuffle version)

Provide an extension procedure for MZVs to arbitrary integer arguments such that

(A) the meromorphic continuation is verified whenever it is defined and

(B) the quasi-shuffle relation is satisfied.

We modify the renormalization problem of multiple zeta values by replacing *quasi-shuffle relation* by *shuffle relation* in (B).

Renormalization problem of MZVs (shuffle version)

Provide an extension procedure for MZVs to **non-positive** integer arguments such that

(A) the meromorphic continuation is verified whenever it is defined and

(B) the **shuffle relation** is satisfied.

Meromorphic continuation

The *multiple zeta function* ζ_n is also defined by the nested series

$$\zeta_n(s_1,\ldots,s_n) := \sum_{m_1>\cdots>m_n>0} \frac{1}{m_1^{s_1}\cdots m_n^{s_n}}.$$

The function $\zeta_n(s_1,\ldots,s_n)$ admits a meromorphic extension to $\mathbb{C}^n$. The subvariety $\mathcal{S}_n$ of singularities is given by

$$\mathcal{S}_n = \left\{ (s_1,\ldots,s_n)\in\mathbb{C}^n\colon \begin{array}{l} s_1 = 1 \text{ or } s_1+s_2 = 2,1,0,-2,-4,\ldots \text{ or} \\ s_1+\cdots+s_j\in\mathbb{Z}_{\leq j} \quad (j=3,4,\ldots,n) \end{array} \right\}.$$

Meromorphic continuation at non-positive integer arguments

- Case $n=1$: For $l\in\mathbb{N}_0$ we have $\zeta_1(-l) = -\frac{B_{l+1}}{l+1}$.
- Case $n=2$: For $k_1,k_2\in\mathbb{N}_0$ with k_1+k_2 odd we have

$$\zeta_2(-k_1,-k_2) = \frac{1}{2}\left(1+\delta_0(k_2)\right)\frac{B_{k_1+k_2+1}}{k_1+k_2+1}.$$

- Case $n\geq 3$: We have $(\mathbb{Z}_{\leq 0})^n \subseteq \mathcal{S}_n$.

Multiple polylogarithms

Definition

For integers $k_1, \dots, k_n$ *multiple polylogarithms* are defined by the iterated sums

$$\mathrm{Li}_{k_1,\dots,k_n}(z) := \sum_{m_1 > \cdots > m_n > 0} \frac{z^{m_1}}{m_1^{k_1} \cdots m_n^{k_n}}$$

where z is a complex number with $|z| < 1$.

Let $k_1 \geq 2$ and $k_2, \dots, k_n \geq 1$. Then one has $\zeta(k_1, \dots, k_n) = \mathrm{Li}_{k_1,\dots,k_n}(1)$. For a tuple $\mathbf{k} := (k_1, \dots, k_n)$ we call

- n the depth of $\mathbf{k}$;
- $|\mathbf{k}| := k_1 + \cdots + k_n$ the weight of $\mathbf{k}$.

The $\mathbb{Q}$-vector space spanned by $\mathcal{M} := \langle \zeta(\mathbf{k}) \colon \mathbf{k} \in \mathbb{N}^n, k_1 > 1, n \in \mathbb{N} \rangle_{\mathbb{Q}}$ is an algebra equipped with two products:

- quasi-shuffle product
- *shuffle product*

Shuffle product

Let $\varphi_1, \ldots, \varphi_p$ be complex-valued differential 1-forms defined on a compact interval. The iterative Chen-integral is defined for real numbers x and y by

$$\int_x^y \varphi_1 \cdots \varphi_p := \int_x^y \varphi_1(t) \int_x^t \varphi_2 \cdots \varphi_p.$$

Theorem (Kontsevich)

Let $\omega_0(t) := \frac{dt}{t}$ and $\omega_1(t) := \frac{dt}{1-t}$. For $k_1, \ldots, k_n \in \mathbb{N}$ we have

$$\mathrm{Li}_{k_1,\ldots,k_n}(z) = \int_0^z \omega_0^{k_1-1}\omega_1 \cdots \omega_0^{k_n-1}\omega_1.$$

Let $Y := \{x_0, x_1\}$ and $\mathfrak{h}^{ш} := \mathbb{Q} \oplus x_0\mathbb{Q}\langle Y\rangle x_1$. For any $u, v \in Y^*$ we define

(i) $\mathbf{1} ш u := u ш \mathbf{1} := u$;

(ii) $au ш bv := a(u ш bv) + b(au ш v)$ for $a, b \in \{x_0, x_1\}$.

For example we have

$$x_0x_1 ш x_0x_1 = 2x_0x_1x_0x_1 + 4x_0^2x_1^2.$$

Lemma (Hoffman 1997)

- *The pair* $(\mathfrak{h}^{ш}, ш)$ *is an algebra.*
- *The map* $\zeta^{ш} : \mathfrak{h}^{ш} \to \mathbb{R}$, $\zeta^{ш}(x_0^{k_1-1}x_1 \cdots x_0^{k_n-1}x_1) := \zeta(k_1, \ldots, k_n)$ *is a morphism of algebras.*

The above example leads to

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1).$$

q-Multiple zeta values

Yesterday, the following model was discussed:

Definition (Ohno, Okuda, Zudilin 2012)

Let $k_1, \dots, k_n \in \mathbb{Z}$ and $|q| < 1$. We define the q-analogue of MZVs (q-MZVs) by

$$\mathfrak{z}_q(k_1, \dots, k_n) := \sum_{m_1 > \cdots > m_n > 0} \frac{q^{m_1}}{[m_1]_q^{k_1} \cdots [m_n]_q^{k_n}},$$

where $[m]_q := \frac{1-q^m}{1-q} = 1 + q + q^2 + \cdots + q^{m-1}$.

- For $k_1 \geq 2$ and $k_2, \dots, k_n \geq 1$ one has

$$\lim_{q \nearrow 1} \mathfrak{z}_q(k_1, \dots, k_n) = \zeta(k_1, \dots, k_n).$$

- The *modified q-MZVs* are defined by

$$\bar{\mathfrak{z}}_q(k_1, \dots, k_n) := (1-q)^{-(k_1 + \cdots + k_n)} \mathfrak{z}_q(k_1, \dots, k_n).$$

Lemma (Castillo, Ebrahimi-Fard, Manchon 2013)

Let $k_1, \ldots, k_n \in \mathbb{Z}$. Then we have

$$\bar{\mathfrak{z}}_q(k_1, \ldots, k_n) = P_q^{k_1}[yP_q^{k_2}[y \cdots P_q^{k_n}[y] \cdots]](q),$$

where $y(t) := \frac{t}{1-t}$ and $P_q[f](t) := \sum_{n \geq 0} f(q^n t)$.

Two important observations: The operator P_q is

- a Rota–Baxter operator (RBO) of weight -1.
- invertible with $P_q^{-1}[f](t) = D_q[f](t) := f(t) - f(qt)$ and D_q satisfies the generalized Leibniz rule, i. e.

$$D_q[fg] = D_q[f]g + fD_q[g] - D_q[f]D_q[g].$$

Therefore, we obtain double q-shuffle relations for *arbitrary integer* arguments. This motivated us to consider *the shuffle version of the renormalization problem of MZVs*.

Shuffle problem

Shuffle problem

- What is the shuffle product for non-positive integer arguments?
- Can we establish a corresponding Hopf algebra for this shuffle product?

Lemma

Let $|z| < 1$ and $k_1, \ldots, k_n \in \mathbb{Z}$. Then we have

$$\mathrm{Li}_{k_1,\ldots,k_n}(z) = J^{k_1}[yJ^{k_2}[y \cdots J^{k_n}[y] \cdots]](z),$$

where $y(z) := \frac{z}{1-z}$ and $J[f](z) := \int_0^z \frac{f(t)}{t}\, dt$.

Two important observations: The operator J is

- a RBO of weight 0 (integration by parts formula).
- invertible with $J^{-1}[f](z) = \delta[f](z) := z\frac{\partial f}{\partial z}(z)$ and δ is a derivation, i.e., it satisfies the Leibniz rule.

Renormalization procedure

1. Construct a family of Hopf algebras in $\lambda \in \mathbb{Q}$ reflecting
   - the q-analogues of MZVs ($\lambda = -1$),
   - the multiple polylogarithms ($\lambda = 0$).
2. Establish a regularization process using deformations of
   - q-analogues of MZVs,
   - multiple polylogarithms.
3. Apply Connes–Kreimer factorization.

Theorem (Connes, Kreimer 2000, Manchon 2008)

Let $(\mathcal{H}, m_{\mathcal{H}}, \Delta)$ be a connected filtered Hopf algebra and $\mathcal{A}$ a commutative unital algebra equipped with a renormalization scheme $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ and corresponding idempotent Rota–Baxter operator π, where $\mathcal{A}_- = \pi(\mathcal{A})$ and $\mathcal{A}_+ = (\mathrm{Id} - \pi)(\mathcal{A})$. Further let $\phi\colon \mathcal{H} \to \mathcal{A}$ be a Hopf algebra character. Then the character ϕ admits a unique decomposition $\phi = \phi_-^{\star(-1)} \star \phi_+$ called algebraic Birkhoff decomposition, in which $\phi_- : \mathcal{H} \to \mathbb{Q} \oplus \mathcal{A}_-$ and $\phi_+ : \mathcal{H} \to \mathcal{A}_+$ are characters.

Word algebraic setting

We introduce a word algebraic setting concerning the shuffle product with non-positive integer arguments:

- $L := \{d, y\}$ be the set of letters.
- L^* free monoid of L with empty word $\mathbf{1}$.
- $\mathbb{Q}\langle L\rangle$ is the free algebra of L.
- $\mathcal{T} := \langle\{wd \colon w \in L^*\}\rangle_{\mathbb{Q}}$ is a subspace of $\mathbb{Q}\langle L\rangle$
- $W := L^*y \cup \{\mathbf{1}\}$ be the set of *admissible words*.
- $\mathcal{H} := \langle W\rangle_{\mathbb{Q}}$ be the algebra spanned by W.

Let $w \in W$. Then we define the

- *weight* $\mathrm{wt}(w)$ given by the number of letters of w;
- *depth* $\mathrm{dpt}(w)$ given by the number of y in w.

Algebra $\mathcal{H}_\lambda$

Let $\lambda \in \mathbb{Q}$. We define the product $⧢_\lambda : \mathbb{Q}\langle L\rangle \otimes \mathbb{Q}\langle L\rangle \to \mathbb{Q}\langle L\rangle$ iteratively by

(P1) $\mathbf{1} ⧢_\lambda w = w ⧢_\lambda \mathbf{1} := w$,

(P2) $yu ⧢_\lambda v = u ⧢_\lambda yv := y(u ⧢_\lambda v)$,

(P3) $\begin{cases} du ⧢_\lambda dv = \frac{1}{\lambda}\left(d(u ⧢_\lambda v) - du ⧢_\lambda v - u ⧢_\lambda dv\right) & \lambda \neq 0, \\ du ⧢_0 dv = d(u ⧢_0 dv) - u ⧢_0 d^2 v & \lambda = 0, \end{cases}$

for any $u, v, w \in L^*$.

Lemma

For $\lambda \in \mathbb{Q}$ the subspace $\mathcal{T}$ is a two-sided ideal of $(\mathbb{Q}\langle L\rangle, ⧢_\lambda)$.

By $\mathcal{L}_0$ we denote the ideal of $(\mathbb{Q}\langle L\rangle, ⧢_0)$ generated by the elements

$$d^k\left(d(u ⧢_0 v) - du ⧢_0 v - u ⧢_0 dv\right), \qquad k \in \mathbb{N}_0; u, v \in L^*.$$

For $\lambda \neq 0$ we define $\mathcal{L}_\lambda := \{0\}$.

Lemma

For $\lambda \in \mathbb{Q}$ the triple $(\mathbb{Q}\langle L\rangle/\mathcal{L}_\lambda, ⧢_\lambda)$ is a $\mathbb{Q}$-algebra.

Coalgebra $\mathcal{H}_\lambda$

We define the coproduct

$$\overline{\Delta}_\lambda : \mathbb{Q}\langle L\rangle \to \mathbb{Q}\langle L\rangle \otimes \mathbb{Q}\langle L\rangle$$

by

(C1) $\overline{\Delta}_\lambda(y) := \mathbf{1} \otimes y + y \otimes \mathbf{1}$,

(C2) $\overline{\Delta}_\lambda(d) := \mathbf{1} \otimes d + d \otimes \mathbf{1} + \lambda d \otimes d$,

which extends uniquely to an algebra morphism (with respect to concatenation) on the free algebra $\mathbb{Q}\langle L\rangle$.

Lemma

For $\lambda \in \mathbb{Q}$ the double $(\mathbb{Q}\langle L\rangle, \overline{\Delta}_\lambda)$ is a cocommmutative coalgebra, and $\mathcal{T}$ and $\mathcal{L}_\lambda$ are coideals of $\mathbb{Q}\langle L\rangle$.

Hopf algebra $\mathcal{H}_\lambda$

Theorem

Let $\lambda \in \mathbb{Q}$ and $\mathcal{H}_\lambda := \mathbb{Q}\langle L\rangle/(\mathcal{T} + \mathcal{L}_\lambda)$. The triple $(\mathcal{H}_\lambda, ⧢_\lambda, \Delta_\lambda)$ is a Hopf algebra with

$$\Delta_\lambda([w]) := \overline{\Delta}_\lambda(w) \mod ((\mathcal{T} + \mathcal{L}_\lambda) \otimes \mathbb{Q}\langle L\rangle + \mathbb{Q}\langle L\rangle \otimes (\mathcal{T} + \mathcal{L}_\lambda))$$

for any word $w \in W$.

Further we obtain a factorization theorem:

Theorem (Shuffle factorization)

Let $\lambda \in \mathbb{Q}$. Then for all $w \in W$ we have

$$⧢_\lambda \circ \Delta_\lambda([w]) = 2^{\mathrm{dpt}([w])}[w].$$

Regularization process

Now need to specify characters Ψ^c and Ψ^q which deform the divergent (q-)MZVs to Laurent series in $\mathbb{Q}[z^{-1}, z]\!]$. For this we define the following maps:

$$\begin{array}{rcccccc} \Psi^c : & (\mathcal{H}_0, ш_0) & \longrightarrow & (\mathbb{Q}[\![t]\!], \cdot) & \longrightarrow & (\mathbb{Q}[z^{-1}, z]\!], \cdot) \\ & [d^{k_1} y \cdots d^{k_n} y] & \longmapsto & \mathrm{Li}_{-k_1, \ldots, -k_n}(t) & \longmapsto & \mathrm{Li}_{-k_1, \ldots, -k_n}(e^z) \end{array}$$

and

$$\begin{array}{rcccccc} \Psi^q : & (\mathcal{H}_{-1}, ш_{-1}) & \longrightarrow & (\mathbb{Q}[\![q]\!], \cdot) & \longrightarrow & (\mathbb{Q}[z^{-1}, z]\!], \cdot) \\ & [d^{k_1} y \cdots d^{k_n} y] & \longmapsto & \bar{\mathfrak{z}}_q(-k_1, \ldots, -k_n) & \longmapsto & \bar{\mathfrak{z}}_{e^z}(-k_1, \ldots, -k_n) \end{array}$$

Theorem

The maps $\Psi^c : (\mathcal{H}_0, ш_0) \to (\mathbb{Q}[z^{-1}, z]\!], \cdot)$ and $\Psi^q : (\mathcal{H}_{-1}, ш_{-1}) \to (\mathbb{Q}[z^{-1}, z]\!], \cdot)$ are well-defined and morphisms of algebras.

Now we are in the position to apply the algebraic Birkhoff decomposition to Ψ^c and Ψ^q. We define for $k_1, \dots, k_n \in \mathbb{N}_0$

- *renormalized MZVs* by

$$\zeta_+(-k_1, \dots, -k_n) := \lim_{z \to 0} \Psi_+^c([d^{k_1} y \cdots d^{k_n} y])(z)$$

- *renormalized q-MZVs* by

$$\mathfrak{z}_+(-k_1, \dots, -k_n) := \lim_{z \to 0} \frac{(-1)^{k_1 + \cdots + k_n}}{z^{k_1 + \cdots + k_n}} \Psi_+^q([d^{k_1} y \cdots d^{k_n} y])(z).$$

Theorem

a) *The renormalization process is compatible with the meromorphic continuation, i.e., ζ_+ coincides with the meromorphic continuation ζ_n whenever it is defined.*

b) *The map ζ_+ satisfies the shuffle product $\sqcup\!\sqcup_0$.*

c) *The map $\mathfrak{z}_+$ is well-defined and for any $\mathbf{k} \in (\mathbb{Z}_{\leq 0})^n$ we have $\zeta_+(\mathbf{k}) = \mathfrak{z}_+(\mathbf{k})$.*

d) *For any $\mathbf{k} \in (\mathbb{Z}_{\leq 0})^n$ we have $\zeta_+(\mathbf{k}) \in \mathbb{Q}$.*

e) *For any character $\psi\colon (\mathcal{H}_0, \sqcup\!\sqcup_0) \to (\mathbb{Q}[z^{-1}, z]\!], \cdot)$ with*

$$\lim_{z\to 0} \psi_+([d^k y])(z) = \zeta_1(-k)$$

for $k \in \mathbb{N}_0$ we have

$$\zeta_+(-k_1, \dots, -k_n) = \lim_{z\to 0} \psi_+([d^{k_1} y \cdots d^{k_n} y])(z)$$

for $k_1, \dots, k_n \in \mathbb{N}_0$.

Numerical examples

$k_1 \diagdown k_2$	0	-1	-2	-3
0	$\frac{1}{4}$	$\frac{1}{24}$	0	$-\frac{1}{240}$
-1	$\frac{1}{12}$	$\frac{1}{144}$	$-\frac{1}{240}$	$-\frac{1}{1440}$
-2	$\frac{1}{72}$	$-\frac{1}{240}$	$-\frac{1}{720}$	$\frac{1}{504}$
-3	$-\frac{1}{120}$	$-\frac{1}{360}$	$\frac{1}{504}$	$\frac{107}{100800}$

Table: The renormalized MZVs $\zeta_+(k_1, k_2)$.

Summary

- Our shuffle renormalization procedure is limited to non-positive integer arguments.
- In the shuffle case the renormalization problem of MZVs has a *unique* solution in contrast to the quasi-shuffle case where we have *infinitely* many solutions.
- Can we expect a double shuffle structure for renormalized MZVs? One has to cope with the following problem:
  - The quasi-shuffle relation

$$\zeta(0)^2 = 2\zeta(0,0) + \zeta(0)$$

    implies $\zeta(0,0) = \frac{3}{8}$ since the meromorphic continuation prescribes $\zeta(0) = -\frac{1}{2}$.
  - From the previous table we know that in the shuffle case $\zeta(0,0) = \frac{1}{4}$.

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Thank you for your attention!