q-Multiple Zeta Values I: from Double Shuffle to Regularisation

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Paths to, from and in renormalisation
Univ. Potsdam – Institut für Mathematik

8-12 February 2016

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Motivation

1. MZVs: regularised shuffle + quasi-shuffle relations

2. Word (quasi-)shuffle Hopf algebras: $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$

$$\Delta(w) := \sum_{uv=w} u \otimes v$$

3. Renormalisation problem for MZVs at non-positive arguments

   - regularisation through a parameter λ: $\zeta_\lambda : \mathcal{H} \to \mathcal{A}$

   - Connes–Kreimer Birkhoff factorisation for Hopf algebra characters

$$\psi = \psi_-^{\star -1} \star \psi_+$$

4. **q-analogues of MZVs**

$$\zeta_q^{\mathrm{BZ}}(k_1, \ldots, k_n) := (1-q)^{k_1+\cdots+k_n} \sum_{m_1>\cdots>m_n>0} \frac{q^{(k_1-1)m_1+\cdots+(k_n-1)m_n}}{(1-q^{m_1})^{k_1}\cdots(1-q^{m_n})^{k_n}}$$

Recall

Multiple Zeta Values (MZVs)

$$\zeta(k_1,\ldots,k_n) := \sum_{m_1>\cdots>m_n>0} \frac{1}{m_1^{k_1}\cdots m_n^{k_n}}$$
$$= \int_{1>t_1>\cdots>t_k>0} \omega_1(t_1)\cdots\omega_k(t_k)$$

$k := k_1+\cdots+k_n$, $\omega_i(t) := \frac{dt}{1-t}$, $i \in \{k_1, k_1+k_2, \ldots, k\}$, and $\omega_i(t) = \frac{dt}{t}$ otherwise

$$\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(b,a) + \zeta(a+b)$$

$$\zeta(a)\zeta(b) = \sum_{i=0}^{a-1} \binom{i+b-1}{b-1} \zeta(b+i, a-i) + \sum_{j=0}^{b-1} \binom{j+a-1}{a-1} \zeta(a+j, b-j)$$

$$\zeta(2)\zeta(2) = 2\zeta(2,2) + \zeta(4) = 4\zeta(3,1) + 2\zeta(2,2)$$

Recall

Shuffle and quasi-shuffle products

Two alphabets:

$$X := \{x_0, x_1\} \qquad Y := \{y_1, y_2, y_3, \ldots\}$$

words and phrases: $X^*,\ \mathbb{Q}\langle X\rangle \qquad Y^*,\ \mathbb{Q}\langle Y\rangle$

$\mathbf{1} ш v = v ш \mathbf{1} = v$ for $v \in X^*$

$$\begin{aligned} x_{k_1} \cdots x_{k_p} ш x_{k_{p+1}} \cdots x_{k_{p+q}} &= x_{k_1}\big(x_{k_2} \cdots x_{k_p} ш x_{k_{p+1}} \cdots x_{k_{p+q}}\big) \\ &\quad + x_{k_{p+1}}\big(x_{k_1} \cdots x_{k_p} ш x_{k_{p+2}} \cdots x_{k_{p+q}}\big) \end{aligned}$$

$\mathbf{1} \,ш\!\!\!-\, u = u \,ш\!\!\!-\, \mathbf{1} = u$ for $u \in Y^*$

$$\begin{aligned} (y_{k_1} \cdots y_{k_p}) \,ш\!\!\!-\, (y_{k_{p+1}} \cdots y_{k_{p+q}}) &= y_{k_1}\big(y_{k_2} \cdots y_{k_p} \,ш\!\!\!-\, y_{k_{p+1}} \cdots y_{k_{p+q}}\big) \\ &\quad + y_{k_{p+1}}\big(y_{k_1} \cdots y_{k_p} \,ш\!\!\!-\, y_{k_{p+2}} \cdots y_{k_{p+q}}\big) \\ &\quad + y_{k_1+k_{p+1}}\big(y_{k_2} \cdots y_{k_p} \,ш\!\!\!-\, y_{k_{p+2}} \cdots y_{k_{p+q}}\big) \end{aligned}$$

Recall

Algebra morphisms

$Y^*_{\text{conv}} := Y^*/y_1Y^*$, i.e., words $y_{n_1}\cdots y_{n_r}$ with $n_1 > 1$

$$\zeta_{⧢}(y_{n_1}\cdots y_{n_r}) := \zeta(n_1,\ldots,n_r)$$

$$\zeta_{⧢}(v \,⧢\, v') = \zeta_{⧢}(v)\zeta_{⧢}(v')$$

$X^*_{\text{conv}} := x_0X^*x_1$, i.e., words $x_0^{n_1-1}x_1\cdots x_0^{n_r-1}x_1$

$$\zeta_{Ш}(x_0^{n_1-1}x_1\cdots x_0^{n_r-1}x_1) := \zeta(n_1,\ldots,n_r)$$

$$\zeta_{Ш}(u \,Ш\, u') = \zeta_{Ш}(u)\zeta_{Ш}(u')$$

Recall

Renormalisation of MZVs

Provide an extension procedure for MZVs to arbitrary integer arguments such that:
(A) the meromorphic continuation is verified whenever it is defined and
(B) the quasi-shuffle relation is satisfied.

Connes–Kreimer Birkhoff decomposition $\psi : \mathcal{H} \to \mathcal{A}$

$$\psi = \psi_-^{\star(-1)} \star \psi_+$$

Regularisation parameter λ

$$\zeta_\lambda(k_1, \ldots, k_n) := \sum_{m_1 > \cdots > m_n > 0} f_{\lambda,k_1}(m_1) \cdots f_{\lambda,k_n}(m_n) \in \mathcal{A}$$

- $f_{\lambda,l}(x) := \frac{\exp(-lx\lambda)}{x^l}$
- $f_{\lambda,l}(x) := \frac{1}{x^{l-\lambda}}$
- $f_{\lambda,l}(x) := \frac{q^{|l|x}}{(1-q^x)^l}$, $\lambda = \log(q)$

$$\lambda \to 0, \;\; f_{\lambda,l}(x) \to f_l(x)$$

q-analogues (regularised) of Multiple Zeta Values

Schlesinger–Zudilin model

$$\zeta_q^{\mathrm{SZ}}(k_1,\ldots,k_n) := \sum_{m_1>\cdots>m_n>0} \frac{q^{k_1m_1+\cdots+k_nm_n}}{[m_1]_q^{k_1}\cdots[m_n]_q^{k_n}} \in \mathbb{Q}[[q]]$$

q-number $[m]_q := \frac{1-q^m}{1-q} = 1+q+q^2+\cdots+q^{m-1}$

modified Schlesinger–Zudilin model

$$\bar{\zeta}_q^{\mathrm{SZ}}(k_1,\ldots,k_n) := \sum_{m_1>\cdots>m_n>0} \frac{q^{k_1m_1+\cdots+k_nm_n}}{(1-q^{m_1})^{k_1}\cdots(1-q^{m_n})^{k_n}}$$

Regularised mod. Schlesinger–Zudilin model

$$\bar{\zeta}_q^{\mathrm{SZ,reg}}(k_1,\ldots,k_n) := \sum_{m_1>\cdots>m_n>0} \frac{q^{|k_1|m_1+\cdots+|k_n|m_n}}{(1-q^{m_1})^{k_1}\cdots(1-q^{m_n})^{k_n}}$$

q-analogues of Multiple Zeta Values

Jackson integral

$$J[f](t) := \int_0^t f(x)d_qx = (1-q)\sum_{n\geq 0} f(q^n t)q^n t$$

$E_q[f](t) := f(qt),\ M_f[g](t) := (fg)(t) = f(t)g(t)$

$$(1-q)\sum_{n\geq 0} f(q^n t)q^n t = (1-q)\sum_{n\geq 0} E_q^n\big[M_{\text{id}}[f]\big](t)$$
$$= (1-q)P_q M_{\text{id}}[f](t)$$

$$P_q[f](t) := \sum_{n\geq 0} E_q^n[f] = f(t) + f(qt) + f(q^2t) + f(q^3t) + \cdots$$

Rota–Baxter map of weight -1

$$P_q[f]P_q[g] = P_q\big[P_q[f]g\big] + P_q\big[fP_q[g]\big] - P_q[fg]$$

q-analogues of Multiple Zeta Values

Replace Riemann by Jackson integral and ...

$$\zeta(k_1, \ldots, k_n) = \int_{1>t_1>\cdots>t_k>0} \omega_1(t_1)\cdots\omega_k(t_k)$$

$k := k_1 + \cdots + k_n$, $\omega_i(t) := \frac{dt}{1-t}$, $i \in \{k_1, k_1+k_2, \ldots, k\}$, and $\omega_i(t) = \frac{dt}{t}$ otherwise

$$\zeta_q^{\mathrm{OOZ}}(k_1, \ldots, k_n) = \int_{q>t_1>\cdots>t_k>0} \omega_1^q(t_1)\cdots\omega_k^q(t_k)$$

$k := k_1 + \cdots + k_n$, $\omega_i^q(t) := \frac{d_qt}{1-t}$, $i \in \{k_1, k_1+k_2, \ldots, k\}$, and $\omega_i^q(t) = \frac{d_qt}{t}$ otherwise

$$\begin{aligned}\zeta_q^{\mathrm{OOZ}}(2) &= (1-q)^2\bar\zeta_q^{\mathrm{OOZ}}(2)\\ &= \int_0^q \frac{d_qt_1}{t_1}\int_0^{t_1}\frac{d_qt_2}{1-t_2} = (1-q)^2 P_qP_q\Big[\frac{t}{1-t}\Big](q) = (1-q)^2\sum_{m>0}\frac{q^m}{(1-q^m)^2}\end{aligned}$$

q-analogues of Multiple Zeta Values

Ohno–Okuda–Zudilin (OOZ) model

$$\zeta_q^{\mathrm{OOZ}}(k_1,\ldots,k_n) = (1-q)^k \underbrace{P_q\left[P_q\left[\cdots P_q\right.\right.}_{k_1}\left[\frac{t}{1-t} \quad \cdots \quad \underbrace{P_q\left[P_q\left[\cdots P_q\right.\right.}_{k_n}\left[\frac{t}{1-t}\right]\right]\right]\cdots\right]\right](q)$$

$$= (1-q)^k \sum_{m_1>\cdots>m_n>0} \frac{q^{m_1}}{(1-q^{m_1})^{k_1}\cdots(1-q^{m_n})^{k_n}}$$

$$= (1-q)^k \bar{\zeta}_q^{\mathrm{OOZ}}(k_1,\ldots,k_n)$$

Recall: P_q is Rota–Baxter map of weight -1

$$P_q[f]P_q[g] = P_q\big[P_q[f]g\big] + P_q\big[f P_q[g]\big] - P_q[fg]$$

Note: operator P_q is invertible

$$P^{-1}[f](t) = D[f](t) := f(t) - f(qt) = (\mathrm{id} - E_q)[f](t)$$

q-shuffle relations I

q-shuffle relation for OOZ model

The q-difference operator D_q satisfies a modified Leibniz rule

$$D_q[fg] = D_q[f]g + fD_q[g] - D_q[f]D_q[g]$$

$$P_q[f]P_q[g] = P_q[P_q[f]g + fP_q[g] - fg]$$

$$D_q[f]D_q[g] = D_q[f]g + fDq[g] - D_q[fg]$$

$$D_q[f]P_q[g] = D_q[fP_q[g]] + D_q[f]g - fg$$

The iterated sum defining $\zeta_q^{\mathrm{OOZ}}(k_1,\ldots,k_n)$ makes sense in $\mathbb{Q}[[q]]$ for any $k_1,\ldots,k_n \in \mathbb{Z}$:

$$\begin{aligned}\zeta_q^{\mathrm{OOZ}}(k_1,\ldots,k_n) &= (1-q)^k \bar{\zeta}_q^{\mathrm{OOZ}}(k_1,\ldots,k_n)\\ &= (1-q)^k P_q^{k_1}\Big[\frac{t}{1-t}\cdots P_q^{k_n}\Big[\frac{t}{1-t}\Big]\cdots\Big](q)\end{aligned}$$

Examples

$$\bar{\zeta}_q^{\mathrm{ooz}}(0) = \sum_{m>0} q^m = \frac{q}{1-q}$$

$$\bar{\zeta}_q^{\mathrm{ooz}}(0,0) = \sum_{m_1>m_2>0} q^{m_1} = \sum_{m>0} (m-1)q^m = \left(\frac{q}{1-q}\right)^2$$

$$\bar{\zeta}_q^{\mathrm{ooz}}(a,0) = \sum_{m_1>m_2>0} \frac{q^{m_1}}{(1-q^{m_1})^a} = \sum_{m>0} \frac{(m-1)q^m}{(1-q^{m_1})^a}$$

$$\bar{\zeta}_q^{\mathrm{ooz}}(1) = \sum_{m>0} \frac{q^m}{1-q^m}$$

$$\bar{\zeta}_q^{\mathrm{ooz}}(-a) = D_q^a[\frac{t}{1-t}](q) = \sum_{m>0} q^m(1-q^m)^a$$

q-shuffle relations II

- Alphabet $\widetilde{X} := \{d, y, p\}$
- $\widetilde{X}^*$ words and good words: $\widetilde{X}^*_{\text{good}} := X^*y$, $dp = pd = \mathbf{1}$, and $k_1, \ldots, k_n \in \mathbb{Z}$

$$w = p^{k_1}y \cdots p^{k_n}y$$

- $\bar{\zeta}^{\text{ooz}}_{q,ш}(p^{n_1}y \cdots p^{n_k}y) := \bar{\zeta}^{\text{ooz}}_q(n_1, \ldots, n_k)$
- The *q-shuffle product* is given on $\mathbb{Q}\langle\widetilde{X}\rangle$ recursively by $\mathbf{1} ш v = v ш \mathbf{1} = v$:

$$\begin{aligned}
(yv) ш u = v ш (yu) &= y(v ш u) \\
pv ш pu &= p(v ш pu) + p(pv ш u) - p(v ш u) \\
dv ш du &= v ш du + dv ш u - d(v ш u) \\
dv ш pu = pu ш dv &= d(v ш pu) + dv ш u - v ш u
\end{aligned}$$

q-shuffle and Euler-type identity

Theorem The q-shuffle product is commutative and associative. For $v, u \in W$ we have:

$$\bar{\zeta}^{\mathrm{OOZ}}_{q,\text{Ш}}(v)\bar{\zeta}^{\mathrm{OOZ}}_{q,\text{Ш}}(u) = \bar{\zeta}^{\mathrm{OOZ}}_{q,\text{Ш}}(v \,\text{Ш}\, u).$$

Euler-type identity:

$$\bar{\zeta}^{\mathrm{OOZ}}_{q}(2)\bar{\zeta}^{\mathrm{OOZ}}_{q}(2) = 2\bar{\zeta}^{\mathrm{OOZ}}_{q}(2,2) + 4\bar{\zeta}^{\mathrm{OOZ}}_{q}(3,1) - 4\bar{\zeta}^{\mathrm{OOZ}}_{q}(2,1) - \bar{\zeta}^{\mathrm{OOZ}}_{q}(3,0) + \bar{\zeta}^{\mathrm{OOZ}}_{q}(2,0)$$

$1 < a \leq b$

$$\bar{\zeta}^{\mathrm{OOZ}}_{q}(-a)\bar{\zeta}^{\mathrm{OOZ}}_{q}(-b) = \sum_{j=0}^{a}\sum_{i=1}^{b-j}(-1)^j\binom{a+b-1-i-j}{a-1}\binom{a}{j}\bar{\zeta}^{\mathrm{OOZ}}_{q}(-j,-i)$$

$$+\sum_{j=0}^{b}\sum_{i=1}^{\max(1,a-j)}(-1)^j\binom{a+b-1-i-j}{b-1}\binom{b}{j}\bar{\zeta}^{\mathrm{OOZ}}_{q}(-j,-i)$$

$$+\sum_{j=1}^{a}(-1)^j\binom{a+b-1-j}{j-1,\,a-j,\,b-j}\bar{\zeta}^{\mathrm{OOZ}}_{q}(-j,0)$$

q-quasi-shuffle relations

$$\bar{\zeta}_q^{\mathrm{ooz}}(a)\bar{\zeta}_q^{\mathrm{ooz}}(b) = \bar{\zeta}_q^{\mathrm{ooz}}(a,b) + \bar{\zeta}_q^{\mathrm{ooz}}(b,a) + \bar{\zeta}_q^{\mathrm{ooz}}(a+b) - \bar{\zeta}_q^{\mathrm{ooz}}(a,b-1) - \bar{\zeta}_q^{\mathrm{ooz}}(b,a-1) - \bar{\zeta}_q^{\mathrm{ooz}}(a+b-1)$$

- Alphabet $\widetilde{Y} := \{z_i : \ i \in \mathbb{Z}\}$ with product $z_i \diamond z_j := z_{i+j}$, $\widetilde{Y}^*$ words
- $\bar{\zeta}_{q,*}^{\mathrm{ooz}}(z_{n_1} \cdots z_{n_k}) := \bar{\zeta}_q^{\mathrm{ooz}}(n_1, \ldots, n_k)$
- Define linear map T on $\mathbb{Q}\langle\widetilde{Y}\rangle$: $T(z_n v) := z_n v - z_{n-1} v$
- The *q-quasi-shuffle product* is given on $\mathbb{Q}\langle\widetilde{Y}\rangle$ through the classical quasi-shuffle product:

$$z_m u * z_n v := z_m\big(u \,ш\, T(z_n v)\big) + z_n\big(T(z_m u) \,ш\, v\big) + (z_{m+n} - z_{m+n-1})(u \,ш\, v)$$

In particular we have:

$$\begin{aligned} z_a * z_b &= z_a(T z_b) + z_b(T z_a) + T z_{a+b} \\ &= z_a z_b + z_a z_b + z_{a+b} - z_a z_{b-1} - z_b z_{a-1} - z_{a+b-1} \end{aligned}$$

Weight drop terms and regularised relations

Theorem The q-quasi-shuffle product is commutative and associative. For $u, v \in \widetilde{Y}^*$ we have:

$$\bar{\zeta}^{\mathrm{ooz}}_{q,*}(u * v) = \bar{\zeta}^{\mathrm{ooz}}_{q,*}(u)\bar{\zeta}^{\mathrm{ooz}}_{q,*}(v).$$

In terms of q-MZVs, the previous example becomes:

$$\zeta^{\mathrm{ooz}}_q(a)\zeta^{\mathrm{ooz}}_q(b) = \zeta^{\mathrm{ooz}}_q(a,b) + \zeta^{\mathrm{ooz}}_q(b,a) + \zeta^{\mathrm{ooz}}_q(a+b) \\ + (1-q)\Big(- \zeta^{\mathrm{ooz}}_q(a,b-1) - \zeta^{\mathrm{ooz}}_q(b,a-1) - \zeta^{\mathrm{ooz}}_q(a+b-1)\Big)$$

In the limit $q \to 1$, the *weight drop terms* disappear $\to$ classical quasi-shuffle relation

$$\begin{aligned} \bar{\zeta}^{\mathrm{ooz}}_q(1)\bar{\zeta}^{\mathrm{ooz}}_q(2) &= \bar{\zeta}^{\mathrm{ooz}}_q(1,2) + \bar{\zeta}^{\mathrm{ooz}}_q(2,1) + \bar{\zeta}^{\mathrm{ooz}}_q(3) - \bar{\zeta}^{\mathrm{ooz}}_q(1,1) - \bar{\zeta}^{\mathrm{ooz}}_q(2,0) - \bar{\zeta}^{\mathrm{ooz}}_q(2) \\ &= \bar{\zeta}^{\mathrm{ooz}}_q(1,2) + 2\bar{\zeta}^{\mathrm{ooz}}_q(2,1) - \bar{\zeta}^{\mathrm{ooz}}_q(2,0) - \bar{\zeta}^{\mathrm{ooz}}_q(1,1) \end{aligned}$$

$$\bar{\zeta}^{\mathrm{ooz}}_q(3) - \bar{\zeta}^{\mathrm{ooz}}_q(2) = \bar{\zeta}^{\mathrm{ooz}}_q(2,1) \quad \longrightarrow \quad \zeta^{\mathrm{ooz}}(3) - (1-q)\zeta^{\mathrm{ooz}}_q(2) = \zeta^{\mathrm{ooz}}_q(2,1)$$

Renormalisation of qMZVs

Remarks: i) The q-quasi-shuffle is not a quasi-shuffle in the sense of Hoffman, i.e., no Hopf algebra structure known. This implies that this model is not suited for renormalisation of q-regularised MZVs using Connes–Kreimer factorisation.

ii) On the other hand, a Hopf algebra can be defined with respect to the q-shuffle product. See Singer's 2nd talk.

iii) Other qMZVs with double q-shuffle (for admissible arguments):

$$\bar{\zeta}_q^{\mathrm{SZ}}(k_1,\ldots,k_n) := \sum_{m_1>\cdots>m_n>0} \frac{q^{k_1m_1+\cdots+k_nm_n}}{(1-q^{m_1})^{k_1}\cdots(1-q^{m_n})^{k_n}}$$

$$\bar{\zeta}_q^{\mathrm{BZ}}(k_1,\ldots,k_n) := \sum_{m_1>\cdots>m_n>0} \frac{q^{(k_1-1)m_1+\cdots+(k_n-1)m_n}}{(1-q^{m_1})^{k_1}\cdots(1-q^{m_n})^{k_n}}$$

Another Rota–Baxter representation for qMZVs

weight -1 RB map $\quad P_q[f](t) := \sum_{n\geq 0} E_q^n[f] = f(t) + f(qt) + f(q^2t) + f(q^3t) + \cdots$

$$P_q[f]P_q[g] = P_q\big[P_q[f]g\big] + P_q\big[fP_q[g]\big] - P_q[fg]$$

weight 1 RB map $\quad \bar{P}_q[f](t) := \sum_{n\geq 1} E_q^n[f] = f(qt) + f(q^2t) + f(q^3t) + \cdots$

$$\bar{P}_q[f]\bar{P}_q[g] = \bar{P}_q\big[\bar{P}_q[f]g\big] + \bar{P}_q\big[f\bar{P}_q[g]\big] + \bar{P}_q[fg]$$

$$\bar{P}_q[f]P_q[g] = \bar{P}_q[fP_q[g]] + P_q[\bar{P}_q[f]g]$$

Double shuffle for other q-MZVs

modified Schlesinger–Zudilin model

$$\bar{\zeta}_q^{\mathrm{SZ}}(k_1,\ldots,k_n) := \sum_{m_1>\cdots>m_n>0} \frac{q^{k_1m_1+\cdots+k_nm_n}}{(1-q^{m_1})^{k_1}\cdots(1-q^{m_n})^{k_n}}$$

$$= \bar{P}_q^{k_1}\big[\frac{t}{1-t}\cdots\bar{P}_q^{k_n}[\frac{t}{1-t}]\cdots\big](1)$$

$$\bar{\zeta}_q^{\mathrm{SZ}}(a)\bar{\zeta}_q^{\mathrm{SZ}}(b) = \bar{\zeta}_q^{\mathrm{SZ}}(a,b) + \bar{\zeta}_q^{\mathrm{SZ}}(b,a) + \bar{\zeta}_q^{\mathrm{SZ}}(a+b)$$

modified Bradley–Zhao model

$$\bar{\zeta}_q^{\mathrm{BZ}}(k_1,\ldots,k_n) := \sum_{m_1>\cdots>m_n>0} \frac{q^{(k_1-1)m_1+\cdots+(k_n-1)m_n}}{(1-q^{m_1})^{k_1}\cdots(1-q^{m_n})^{k_n}}$$

$$= \bar{P}_q^{k_1-1}P_q\big[\frac{t}{1-t}\cdots\bar{P}_q^{k_n-1}P_q[\frac{t}{1-t}]\cdots\big](1)$$

$$\bar{\zeta}_q^{\mathrm{BZ}}(a)\bar{\zeta}_q^{\mathrm{BZ}}(b) = \bar{\zeta}_q^{\mathrm{BZ}}(a,b) + \bar{\zeta}_q^{\mathrm{BZ}}(b,a) + \bar{\zeta}_q^{\mathrm{BZ}}(a+b) + \bar{\zeta}_q^{\mathrm{BZ}}(a+b-1).$$

Regularised mod. Schlesinger–Zudilin model

$$\bar{\zeta}_q^{\mathrm{SZ,reg}}(k_1,\ldots,k_n) := \sum_{m_1>\cdots>m_n>0} \frac{q^{|k_1|m_1+\cdots+|k_n|m_n}}{(1-q^{m_1})^{k_1}\cdots(1-q^{m_n})^{k_n}}$$

defined for $k_1,\ldots,k_n$ strictly negative.

1) (q-)quasi-shuffle: for $a,b\in\mathbb{N}$.

$$\bar{\zeta}_q^{\mathrm{SZ,reg}}(-a)\bar{\zeta}_q^{\mathrm{SZ,reg}}(-b) = \sum_{m>0}\left(q^m(1-q^m)\right)^a \sum_{n>0}\left(q^n(1-q^n)\right)^b$$

$$= \sum_{m>n>0}\left(q^m(1-q^m)\right)^a\left(q^n(1-q^n)\right)^b + \sum_{n>m>0}\left(q^n(1-q^n)\right)^b\left(q^m(1-q^m)\right)^a$$

$$+ \sum_{m>0}\left(q^m(1-q^m)\right)^{a+b}$$

$$= \bar{\zeta}_q^{\mathrm{SZ,reg}}(-a,-b) + \bar{\zeta}_q^{\mathrm{SZ,reg}}(-b,-a) + \bar{\zeta}_q^{\mathrm{SZ,reg}}(-a-b)$$

2) no q-shuffle representation