# Renormalisation & resurgent transseries in quantum field theory

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Paths to, from and in renormalisation
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# Outline

# Basic ingredients

1. finite lattice: $\Gamma = \varepsilon\mathbb{Z}^d / L\mathbb{Z}^d$ with $\frac{L}{2\varepsilon} \in \mathbb{N}$ (discrete torus)
2. real scalar field $\phi$: $\Gamma \to \mathbb{R}$

**Euclidean action of $\phi^4$ theory**

$$S_\Gamma(\phi, \lambda) = \frac{1}{2}\int_\Gamma \phi[-\Delta + m^2]\phi + \lambda \int_\Gamma \phi^4 \qquad (1)$$

where $\int_\Gamma F := \varepsilon^d \sum_{x\in\Gamma} F(x)$ for $F \in \mathbb{R}^\Gamma$ and

$$-\Delta\phi(x) = \frac{1}{\varepsilon^2}\sum_{j=1}^{d}[2\phi(x) - \phi(x+\varepsilon e_j) - \phi(x - \varepsilon e_j)]$$

lattice Laplacian

# Partition function

## Partition function as path integral:

$$\mathscr{Z}_\Gamma(J,\lambda) = \int \mathcal{D}_\Gamma\phi\; e^{-S_\Gamma(\phi,\lambda)+\int_\Gamma J\cdot\phi} \tag{2}$$

where $J \in \mathbb{R}^\Gamma$ external field and

Lebesgue measure in $\mathbb{R}^{|\Gamma|}$ : $\qquad \mathcal{D}_\Gamma\phi = \prod_{x\in\Gamma} d\phi(x)$

All quantities obtained from $\mathscr{Z}_\Gamma(J,\lambda)$, eg

## Correlators

$$\langle\phi(x_1)\dots\phi(x_n)\rangle_\lambda = \frac{1}{\mathscr{Z}_\Gamma(0,\lambda)}\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\mathscr{Z}_\Gamma(J,\lambda)\Big|_{J=0}$$

# Coupling dependence

Continuum limit: $\varepsilon \to 0$ and/or $L \to \infty$

## Question

Given continuum limit exists, does

$$\lambda \mapsto \frac{\mathscr{Z}_\Gamma(J,\lambda)}{\mathscr{Z}_\Gamma(0,\lambda)} = \frac{1}{\mathscr{Z}_\Gamma(0,\lambda)} \int \mathcal{D}_\Gamma\phi \; e^{-S_\Gamma(\phi,0)-\lambda\int_\Gamma \phi^4 + \int_\Gamma J\cdot\phi} \tag{3}$$

belong to Écalle's class of analysable functions in this limit?

1. Is there a valid transseries representation?
2. If so, what is it and is it accelero-summable?

# Analysable functions: 'field with no escape'

Class of analysable functions is stable under

- algebraic operations of a field (like $\mathbb{C}$)
- composition and inversion (if injective)
- integration and differentiation

## Generators

Take $\mathbb{C}$-linear span of

$$1, z, e^z, \log z$$

and perform all of the above operations.

example: $\int \frac{1}{\log z} = z \sum_{k \geq 0} \frac{k!}{(\log z)^{k+1}}$, $\int z^{-1} e^z = e^z \sum_{k \geq 0} k! z^{-k-1}$, $\int e^{e^z} = \dots$

# Transseries

## Grid-based transseries

formal series of the form

$$\sum_{l_1 \geq \alpha_1} \cdots \sum_{l_k \geq \alpha_k} c_{(l_1,\dots,l_k)} \mathfrak{m}_1^{l_1} \cdots \mathfrak{m}_k^{l_k} \qquad (\alpha_j \in \mathbb{Z})$$

with transmonomials $\mathfrak{m}_1, \dots, \mathfrak{m}_k$, no convergence required

## Group of transmonomials: examples

$$z^{-1}, e^z, z^4 e^{-z}, e^{e^z \sum_{j\geq 0} z^{-j}}, e^{-z+z^2}, \log z, \log \circ \log z, \dots$$

## Accelero-summation of height-one transseries

formal transseries $\xrightarrow{\mathcal{B}}$ convergent transseries $\xrightarrow{\mathcal{L}}$ analysable fct

# Semi-classical expansion I

Rescaling: $\varphi := \lambda^{\frac{1}{2}}\phi \qquad \mathcal{J} := \lambda^{-\frac{1}{2}} J \qquad \mathcal{D}_\Gamma\varphi = \lambda^{\frac{|\Gamma|}{2}}\mathcal{D}_\Gamma\phi$

## partition function, rescaled

$$\frac{\mathscr{Z}_\Gamma(\lambda^{\frac{1}{2}}\mathcal{J},\lambda)}{\mathscr{Z}_\Gamma(0,\lambda)} = \frac{1}{\mathscr{Z}_\Gamma(0,\lambda)}\int \mathcal{D}_\Gamma\varphi \; e^{-\frac{1}{\lambda}S_\Gamma(\varphi,1)+\int_\Gamma \mathcal{J}\cdot\varphi} \tag{4}$$

## Semi-classical expansion around critical points

$$\frac{\mathscr{Z}_\Gamma(\lambda^{\frac{1}{2}}\mathcal{J},\lambda)}{\mathscr{Z}_\Gamma(0,\lambda)} \cong \underbrace{\sum_{\varphi_c} e^{-\frac{1}{\lambda}S_\Gamma(\varphi_c,1)} F_{\varphi_c}(\mathcal{J},\lambda)}_{\text{transseries}} \in \sum_{\varphi_c} e^{-\frac{1}{\lambda}S_\Gamma(\varphi_c,1)}\mathbb{C}[[\lambda]]$$

(fixed $\mathcal{J}$) connection to transseries: $z = \lambda^{-1}$

# Topical transseries ansätze

Currently used ansätze of the form

**Height-1, depth-1 transseries**

$$f = \sum_{\sigma \in \mathbb{N}_0^r} z^{c\cdot\sigma} e^{-(b\cdot\sigma)z} P_\sigma(\log z) \sum_{s \geq 0} c_{(\sigma,s)} z^{-s}$$

$c, b \in \mathbb{C}^r$, $P_\sigma(\log z) \in \mathbb{C}[\log z]$ polynomial, $z = (\text{coupling})^{-1}$

in quantum mechanics, toy model and SUSY QFTs, toy model and SUSY string theories

**Sectors of the transseries**

subseries for fixed $\sigma \in \mathbb{N}_0^r$:

$$z^{c\cdot\sigma} e^{-(b\cdot\sigma)z} P_\sigma(\log z) \sum_{s \geq 0} c_{(\sigma,s)} z^{-s}.$$

# Perturbation theory (path integral approach)

partition function revisited

$$\frac{\mathscr{Z}_\Gamma(J,\lambda)}{\mathscr{Z}_\Gamma(0,\lambda)} = \frac{1}{\mathscr{Z}_\Gamma(0,\lambda)} \int \mathcal{D}_\Gamma\phi \; e^{-S_\Gamma(\phi,0)-\lambda\int_\Gamma \phi^4+\int_\Gamma J\cdot\phi}$$

## Idea of perturbation theory

expand interaction exponential in $\lambda$

$$e^{-\lambda\int_\Gamma \phi^4} e^{\int_\Gamma J\cdot\phi} = \sum_{s\geq 0}\left(-\int_\Gamma \phi^4\right)^s e^{\int_\Gamma J\cdot\phi}\lambda^s$$

generate polynomials in $\phi$ and compute Gaussian expectations

# The need for renormalisation

**Problem: divergent expectations**

already at order $\mathcal{O}(\lambda)$,

$$\left| \int d\mu_\Gamma(\phi)\phi(x)^4 \right| \to \infty \qquad \text{as } \varepsilon \to 0 \text{ or } L \to \infty,$$

where $d\mu_\Gamma(\phi) = \mathscr{Z}_\Gamma(0,0)^{-1}\mathcal{D}_\Gamma\phi \; e^{-S_\Gamma(\phi,0)}$

**Solution in $d = 2$: Wick ordering**

$$:\phi(x)^4: \; = \phi(x)^4 - 6C_\Gamma(x,x)\phi(x)^2 + 3C_\Gamma(x,x)^2$$

with $C_\Gamma(x,y) = \langle \phi(x)\phi(y)\rangle_0 = \int d\mu_\Gamma(\phi)\phi(x)\phi(y)$, then

$$\int d\mu_\Gamma(\phi) :\phi(x)^4: \; = 0 \qquad \forall \varepsilon > 0$$

# Superrenormalsation I: Z factor in $d = 2$

**(super)renormalised Euclidean action ($d = 2$)**

$$\mathscr{R}_2[S_\Gamma](\phi,\lambda) = \frac{1}{2}\int_\Gamma \phi[-\Delta + m^2]\phi + \lambda \int_\Gamma :\phi^4:$$

In physics: replace $m^2$ by $m^2 Z_m(\lambda) := m^2(1 + c_1\lambda)$ to obtain

**mass-corrected Euclidean action ($d = 2$) ($\sim$ Wick ordered)**

$$\mathscr{R}_2[S_\Gamma](\phi,\lambda) = \frac{1}{2}\int_\Gamma \phi[-\Delta + m^2 Z_m(\lambda)]\phi + \lambda \int_\Gamma \phi^4$$

renormalisation Z factor:

$$Z_m(\lambda) = 1 + c_1\lambda = 1 - 12m^{-2}C_\Gamma(0,0)\lambda$$

where $C_\Gamma(0,0) = C_\Gamma(x,x)$ constant ($\to \infty$ in continuum limit)

# Superrenormalsation II: Z factor in $d = 3$

- Wick ordering not sufficient in $d = 3$!
- One additional counterterm necessary.

**Mass-renormalisation Z factor in spacetime dimension $d = 3$:**

$$Z_m(\lambda) = 1 + c_1\lambda + c_2\lambda^2$$

mass-corrected Euclidean action ($d = 3$)

$$\mathscr{R}_3[S_\Gamma](\phi,\lambda) = \frac{1}{2}\int_\Gamma \phi[-\Delta + m^2 Z_m(\lambda)]\phi + \lambda\int_\Gamma \phi^4$$

# Renormalisation in $d = 4$

Jump in complexity:

**General form of (super)renormalised action in dimension $d$**

$$\mathscr{R}_d[S_\Gamma](\phi,\lambda) = \frac{1}{2}\int_\Gamma \phi[Z(\lambda)(-\Delta) + m^2 Z_m(\lambda)]\phi + \lambda Z_v(\lambda)\int_\Gamma \phi^4$$

1. $d = 2$: $Z(\lambda) = 1$ , $Z_m(\lambda) = 1 + c_1\lambda$ , $Z_v(\lambda) = 1$
2. $d = 3$: $Z(\lambda) = 1$ , $Z_m(\lambda) = 1 + c_1\lambda + c_2\lambda^2$ , $Z_v(\lambda) = 1$
3. $d = 4$: asymptotic power series

$$Z(\lambda) = \sum_{s\geq 0} a_s\lambda^s, \quad Z_m(\lambda) = \sum_{s\geq 0} c_s\lambda^s, \quad Z_v(\lambda) = \sum_{s\geq 0} b_s\lambda^s$$

# Semi-classical expansion II, renormalised case

Rescaling hopeless for $d = 4$:

**partition function**

$$\frac{\mathscr{Z}_\Gamma(J,\lambda)}{\mathscr{Z}_\Gamma(0,\lambda)} = \frac{1}{\mathscr{Z}_\Gamma(0,\lambda)} \int \mathcal{D}_\Gamma\phi \; e^{-\mathscr{R}_d[S_\Gamma](\phi,\lambda)+\int_\Gamma J\cdot\phi} \tag{5}$$

**Semi-classical expansion around critical points**

$$\frac{\mathscr{Z}_\Gamma(J,\lambda)}{\mathscr{Z}_\Gamma(0,\lambda)} \cong \underbrace{\sum_{\phi_c} e^{-\mathscr{R}_d[S_\Gamma](\phi_c,\lambda)} F_{\phi_c}(J,\lambda)}_{\text{transseries?}} \in \text{transseries class ?}$$

(fixed $\mathcal{J}$)

# Dyson-Schwinger equations I: Schwinger's approach

## Identities for correlators

Idea: derive from

$$\int \mathcal{D}_\Gamma \phi \; \frac{\delta}{\delta\phi(x)} \left[ e^{-S_\Gamma(\phi,\lambda)} \phi(x_1) \cdots \phi(x_n) \right] = 0$$

identities for correlators.

## Example $n = 1$

$$\left\langle \frac{\delta S_\Gamma(\phi,\lambda)}{\delta\phi(x)} \phi(y) + \delta_\Gamma(x,y) \right\rangle = 0,$$

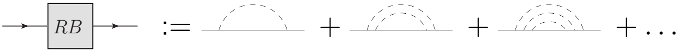
where $\delta_\Gamma(x,y) = \varepsilon^{-d}\delta_{x,y}$ and

$$\frac{\delta S_\Gamma(\phi,\lambda)}{\delta\phi(x)} = (-\Delta + m^2)\phi(x) + 4\lambda\phi(x)^3$$

# Dyson-Schwinger equations II: Dyson's approach

Identities from self-similiarity of Feynman diagram series.

- example: rainbow approximation in Yukawa theory




stands for perturbative series

$$\Sigma = a \int \mathcal{K} + a^2 \int \mathcal{K} \int \mathcal{K} + a^3 \int \mathcal{K} \int \mathcal{K} \int \mathcal{K} + \ldots$$

$a$ coupling, $K$ integral kernel of Feynman integral :

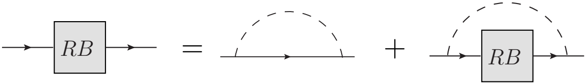
$$= \int \mathcal{K} = \int \frac{d^4 k}{2\pi^2} \left\{ \frac{1}{k^2 (q-k)^2} - \frac{1}{k^2 (\tilde{q} - k)^2} \right\}$$

# Self-similiarity of Feynman diagram series

- rainbow Dyson-Schwinger equation

$$\Sigma = a \int \mathcal{K}(1 + a \int \mathcal{K} + a^2 \int \mathcal{K} \int \mathcal{K} + \dots) = a \int \mathcal{K}(1 + \Sigma)$$

- diagrammatically:




concretely, massless in $d = 4$

$$\Sigma(q^2, a) = a \int \frac{d^4 k}{2\pi^2} \left\{ \frac{1}{k^2 (q-k)^2} - \frac{1}{k^2 (\tilde{q}-k)^2} \right\} \left[ 1 + \Sigma(q^2, a) \right]$$

# Kilroy Dyson-Schwinger equation in Yukawa theory

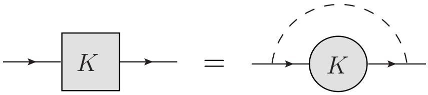
## Kilroy DSE for self-energy

$$\Sigma(q^2, a) = a \int d^4k \, \mathcal{K}(k, q, \tilde{q}) \left[1 - \Sigma(q^2, a)\right]^{-1}$$

where

$$\mathcal{K}(k, q, \tilde{q}) = \frac{1}{2\pi^2} \left\{ \frac{1}{k^2(q-k)^2} - \frac{1}{k^2(\tilde{q}-k)^2} \right\},$$

diagrammatically:



# Terminology: self-energy & anomalous dimension

## Self-energy in $(\phi^4)_d$

$$\langle\phi(x)\phi(y)\rangle = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik(x-y)}}{k^2 + m^2 + \Sigma(k^2, \lambda)}$$

## anomalous dimension

$$\gamma(\lambda) = k^2 \frac{\partial}{\partial k^2} \Sigma(k^2, \lambda)\Big|_{k^2=1}$$

## Question

known perturbatively, but: transseries representation of $\gamma(\lambda)$?

# Result Ia: fixed-point equation from Kilroy DSE

back to Yukawa theory:

**DSE for anomalous dimension**

$$\gamma(a) = C_0 a + C_1 a\gamma(a) + a \sum_{r\geq 2} \sum_{n\geq r} (\gamma_\bullet^{\star r})_n(a),$$

where

$$(\gamma_\bullet^{\star r})_n(a) := \sum_{n_1+\ldots+n_r=n} \frac{\gamma_{n_1}(a)}{n_1!} \cdots \frac{\gamma_{n_r}(a)}{n_r!}$$

and

$$\gamma_n(a) = (\gamma(a)[2a\partial_a - 1])^{n-1}\gamma(a)$$

hence: rhs of above DSE has an infinite # of differential operators

# Result Ib: ansatz wrong

## Transseries ansatz: an ill fit

plug

$$\gamma(z) = \sum_{\sigma \geq 0} \sum_{s \geq 0} c_{(\sigma,s)} z^{\sigma c} e^{-\sigma(b_1 z + b_2 z^2)} z^{-s}$$

into fixed point equation and get $c_{(\sigma,s)} = 0$ for all $\sigma \geq 1$.

## Kilroy ODE from DSE

insert ansatz with $b_1 z + \ldots + b_m z^m$ upstairs into

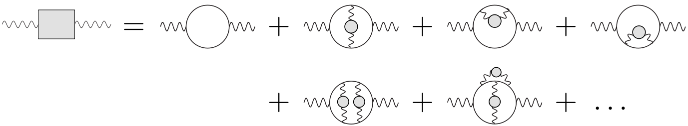
$$\gamma(a) + \gamma(a)[2a\partial_a - 1]\gamma(a) = a/2$$

and find the same for all $m \geq 1$.

Logarithmic transmonomials expedient? No.

# Result II: photon DSE in QED

self-consistent DSE for photon propagator



leads to

$$\gamma = \alpha A_0 + \sum_{\ell \geq 1} \alpha^{\ell+1} \sum_{r_1 \geq 0, n_1 \geq r_1} \cdots \sum_{r_\ell \geq 0, n_\ell \geq r_\ell} C_{(n_1, \ldots, n_\ell)} (\gamma_\bullet^{\star r_1})_{n_1} \cdots (\gamma_\bullet^{\star r_\ell})_{n_\ell}$$

for anomalous dimension. Result: transseries ansatz yet again ill fit

# Conclusion

1. renormalisation complicates matters by rendering coupling dependence of action nontrivial
2. for renormalised quantum field theories, we (probably) need fancier transseries ansätze,

$$\gamma(z) = \sum_{(\sigma,t,j)\geq(0,0,0)} c_{(\sigma,t,j)} z^{-\sigma c} e^{-\sigma(b_1 z + \ldots + b_m z^m)} z^{-t} (\log z)^j$$

   is not elaborate enough
3. future transseries may involve superexponentials $\mathfrak{m} = e^{-e^z}$.