Wonderful Renormalization

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Introduction

QFT in position space / Causal perturbation theory

- Stueckelberg, Bogoliubov, Shirkov (late 50's): Axiomatic approach to S-matrix,

$$S = 1 + \sum_{n>1} T_n.$$

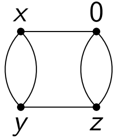
- Epstein and Glaser ('73): Renormalization of S translates into an extension (splitting) problem for distributions.
- Simplified version by Stora (ca.'00), used in QFT on curved spacetimes.

Introduction

- Bergbauer, Brunetti, Kreimer ('10): Version for single graphs.

Example (Euclidean ϕ_4^4-theory)

G



Feynman rules $\Phi : G \longmapsto \int \omega_G = \int dxdydz \frac{1}{(x-y)^4 (y-z)^2 z^4 x^2}$

What is $\int \omega_G$?

- Easy answer: ∞.
- Tricky answer: Find renormalized value ...

Introduction

Idea (Atiyah; Axelrod, Singer): Use a smooth model to arrange the divergences in a "nice" way, renormalize on this model, then push the result back to original spacetime.

Definition

Let $\mathcal{A} = \{A_1, \ldots, A_k\}$ be a family of smooth subvarities in an algebraic variety X. A smooth model is a smooth variety Y together with a proper, surjective map $\beta : Y \to X$, such that $\mathcal{E} := \beta^{-1}(\cup_{A \in \mathcal{A}} A)$ is a normal crossing divisor and $\beta_{|Y \setminus \mathcal{E}}$ a diffeomorphism.

Wonderful models

Such smooth models are given by the wonderful model construction by DeConcini and Procesi. Idea is based on Fulton and MacPherson's "Compactification of Configuration Spaces":

The configuration space of n-points in an algebraic variety X is

$$C_n(X) = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for all } i \neq j\}.$$

Fulton and MacPherson construct its compactification $X[n]$ by a sequence of blow-ups along the (strict transforms) of diagonals of increasing dimension. A limiting point in $X[n] \setminus C_n(X)$ is encoded by a nested set of diagonals.

Wonderful models

Definition

Let $\mathcal{A}$ be a linear arrangement in a vector space X. The wonderful model $(Y_{\mathcal{A}}, \beta)$ is defined as follows: The graph of the map

$$\pi_{\mathcal{A}} : X \setminus \bigcup_{A \in \mathcal{A}} A \longrightarrow \prod_{A \in \mathcal{A}} \mathbb{P}(X/A)$$

is locally closed in $X \times \prod_{A \in \mathcal{A}} \mathbb{P}(X/A)$. Define $Y_{\mathcal{A}}$ as its closure and $\beta : Y_{\mathcal{A}} \to X$ as the projection onto the first factor.

- An explicit construction is given by a sequence of blowups along (strict transforms of) elements of a building set $\mathcal{B} \subseteq \mathcal{A}$, giving local charts $(U_{\underline{i}}, \kappa_{\underline{i}})$, $\underline{i} = (\mathcal{N}, B)$, where $\mathcal{N}$ is a nested set of elements of $\mathcal{B}$ and B an adapted, marked basis of X.
- $\mathcal{B}$ controls the number of irreducible components of $\mathcal{E} \subseteq Y_{\mathcal{B}}$, while the $\mathcal{B}$-nested sets describe a stratification of $\mathcal{E}$.

Graphs and arrangements

Feichtner ('05): These notions can all be defined combinatorially!
Either in terms of the intersection lattice of $\mathcal{A}$,

$$\mathcal{L}_{\mathcal{A}} := \left\{ \{A_1 \cap \dots \cap A_k \mid A_i \in \mathcal{A}\}, \supseteq \right\},$$

or, in our case, using the poset of divergent subgraphs of G.

Definition

Let $G = (V, E)$ be a graph.

- Its superficial degree of divergence is defined by $s(G) = dh_1(G) - 2|E|$ ($d =$ dim. of spacetime). G is called at most logarithmic if $s(g) \leq 0$ holds for all $g \subseteq G$.
- The divergent poset of G is defined as

$$\mathcal{D}_G := \left\{ \{g \subseteq G \mid s(g) \leq 0\}, \subseteq \right\}.$$

Now consider the following Feynman rules:
Let G be a connected graph. Orient G and choose a spanning tree $t \subseteq G$.
The Feynman rules map Φ sends G to the pair $(X_G, \tilde{v}_G)$ of a chain $X_G = (\mathbb{R}^d)^{E(t)}$ and a form defined by the rational function

$$v_G : x \longmapsto \prod_{e \in E(G)} y_e^{-\frac{d}{2}}, \quad y_e = \begin{cases} x_e & \text{if } e \in E(t) \\ \sum_{e' \in E(t_e)} \sigma_t(e') x_{e'} & \text{else.} \end{cases}$$

Here t_e is the unique path in t connecting the source and target vertices of e and $\sigma_t : E(t) \to \{\pm 1\}$ given by the orientation on G.

Graphs and arrangements

We avoid the infrared problem - $v_G \notin L^1(X_G)$ - by viewing v_G as (the kernel of) a distribution on X_G. On the other hand, the ultraviolet problem - $v_G \notin L^1_{\mathrm{loc}}(X_G)$ - is characterized by the following

Proposition

Let G be at most logarithmic.

- v_G *defines a distribution on* $X_G \setminus \bigcup_{g \in \mathcal{D}_G} A_g$, *where* $A_g := \{y_e = 0 \mid e \in E(g)\} \subseteq X_G$.
- $\mathcal{D}_G$ *is a graded (distributive) lattice with join and meet operations given by*

$$g \vee h := g \cup h$$
$$g \wedge h := g \cap h.$$

Wonderful combinatorics

Definition

Let $\mathcal{L}$ be a lattice. $\mathcal{B} \subseteq \mathcal{L}$ is a building set for $\mathcal{L}$ if

- for all $p \in \mathcal{L}_{>\hat{0}}$ and $\{q_1, \ldots, q_k\} = \max \mathcal{B}_{\leq p}$ there is an isomorphism of posets

$$\varphi_A : \prod_{i=1}^{k} [\hat{0}, q_i] \longrightarrow [\hat{0}, p]$$

  with $\varphi_p(\hat{0}, \ldots, q_j, \ldots, \hat{0}) = q_j$ for $j = 1, \ldots, k$.
- the ranking function on $\mathcal{L}$ satisfies

$$r(p) = \sum_{i=1}^{k} r(q_i).$$

In our case r is given by $\operatorname{codim}(A_g) = d(|E(g)| - h_1(g))$.

Wonderful combinatorics

Definition

Let $\mathcal{B}$ be a building set in $\mathcal{L}$. A subset $\mathcal{N} \subseteq \mathcal{B}$ is $\mathcal{B}$-nested if for all subsets $\{p_1, \ldots, p_k\} \subseteq \mathcal{N}$ of pairwise incomparable elements their join $p_1 \vee \cdots \vee p_k$ exists (in $\mathcal{L}$) and does not belong to $\mathcal{B}$.

Definition

- A basis b of X is adapted to $\mathcal{N}$ if for all $A \in \mathcal{N}$ the set $b \cap A$ generates $A \iff b$ is given by the edges of an adapted spanning tree, i.e. $t \subseteq G$ such that $t \cap g$ is spanning for all $g \in \mathcal{N}$.
- A marking of b is for every $A \in \mathcal{N}$ the choice of an element $b_A \in b \cap A \iff$ for every $g \in \mathcal{N}$ a choice $b_g \in \{b_e\}_{e \in E(t \cap g)}$.

Wonderful renormalization

Let $(Y_{\mathcal{B}}, \beta)$ be a wonderful model for a building set $\mathcal{B} \subseteq \mathcal{D} = \mathcal{D}_G$ and $v = v_G$ the Feynman distribution associated to a graph G.

Proposition

In local coordinates on $U_{\underline{i}}, \underline{i} = (\mathcal{N}, B)$, the pullback of $\tilde{v}^s = v^s|dx|$ ($s =$ regularization parameter) along β is a density on Y given by

$$(\omega^s)_{\underline{i}} := (\beta^* \tilde{v}^s)_{\underline{i}} = f_{\underline{i}}^s \prod_{g \in \mathcal{N}} u_g(s, \cdot)|dx|,$$

$$u_g(s, x) = |x_g|^{-1+r(g)(s-1)}, x_g \text{ marked.}$$

The map $f_{\underline{i}} : \kappa_{\underline{i}}(U_{\underline{i}}) \longrightarrow \mathbb{R}$ is in $L^1_{loc}(\kappa_{\underline{i}}(U_{\underline{i}}))$ and smooth in the marked variables x_g, $g \in \mathcal{N}$.

Wonderful renormalization

The next step is to study the Laurent expansion of ω^s. To formulate this we need a local version of graph contraction.

Definition

Let $g \subseteq G$ and $\mathcal{N}$ be nested. The contraction relative to $\mathcal{N}$ is defined as

$$g//\mathcal{N} := \begin{cases} g/(\bigcup_{\gamma\in\mathcal{N}_{<g}} \gamma) & \text{if } g \in \mathcal{N}, \\ g/(g \cap \bigcup_{\gamma\in\mathcal{N},\gamma\cap g<g} \gamma) & \text{else.} \end{cases}$$

For $\mathcal{J} \subseteq \mathcal{N}$ the poset $(\mathcal{N}//\mathcal{J}, \sqsubseteq)$ is given by the underlying set

$$\mathcal{N}//\mathcal{J} := \{g//\mathcal{J} \mid g \in \mathcal{N}\},$$

partially ordered by inclusion (in general $\sqsubseteq \neq \subseteq$!).

Wonderful renormalization

Theorem

- *The Laurent expansion of* ω^s *at* $s = 1$ *has a pole of order* N *where* N *is the cardinality of the largest* $\mathcal{B}$*-nested set.*
- *The coefficients* $\tilde{a}_k$ *in the principal part of the Laurent expansion*

$$\omega^s = \sum_{-N \leq k} \tilde{a}_k (s-1)^k$$

are densities with supp $\tilde{a}_k = \bigcup_{|\mathcal{N}|=-k} \mathcal{E}_{\mathcal{N}}$ *for* $k < 0$*.*

- *Consider the minimal building set* $I(\mathcal{D}) \subseteq \mathcal{D}$*. Assume* $G \in I(\mathcal{D})$*. Let* N *be the cardinality of a maximal nested set and denote by* χ *the constant function on the wonderful model* $Y_{I(\mathcal{D})}$*. Then*

$$\langle \tilde{a}_{-N} | \chi \rangle = \sum_{|\mathcal{M}|=N} \prod_{\gamma \in \mathcal{M}} \mathcal{P}(\gamma // \mathcal{M}).$$

Wonderful renormalization

Definition ("Local subtraction at fixed conditions")

In every chart $U_{\underline{i}}$ let $\nu = \{\nu_g^{\underline{i}}\}_{g\in\mathcal{N}}$ denote a collection of smooth functions on $\kappa(U_{\underline{i}})$, each $\nu_g^{\underline{i}}$ depending only on the coordinates x_e with $e \in E(t) \cap E(g \setminus \mathcal{N}_{<g})$, satisfying $\nu_g^{\underline{i}}|_{x_g=0} = 1$ and compactly supported in all other directions. For $u \in \mathcal{D}'(\mathbb{R} \setminus \{0\})$ and $\mu \in \mathcal{D}([-1,1])$ let $r_\mu[u] \in \mathcal{D}'(\mathbb{R})$ denote the extended distribution

$$r_\mu[u] : \varphi \mapsto \langle u|\varphi\rangle - \langle u|\varphi(0)\mu\rangle.$$

The extension of ω^s is defined by

$$\begin{aligned} R_\nu[\omega^s] \overset{loc.}{=} R_\nu^{\underline{i}}[f_{\underline{i}}^s \prod_{g\in\mathcal{N}} u_g(s)|dx|] &:= f_{\underline{i}}^s \prod_{g\in\mathcal{N}} r_{\nu_g^{\underline{i}}}[u_g(s)]|dx| \\ &=: \sum_{\mathcal{J}\subseteq\mathcal{N}} (-1)^{|\mathcal{J}|}\nu_{\mathcal{J}}^{\underline{i}} \cdot (\omega_{\underline{i}}^s)_{\mathcal{E}_{\mathcal{J}}}. \end{aligned}$$

Wonderful renormalization

Theorem

- *$R_\nu[\omega^s]$ defines a density-valued holomorphic function in a neighborhood of $s = 1$.*
- *Define the renormalized Feynman rules by the map*

$$\Phi_R : G \longmapsto (X_G, \mathscr{R}[v_G])$$

with $\mathscr{R}[v_G] := \beta_* R_\nu[\omega^s]_{|s=1}$ and evaluation on $\varphi \in \mathcal{D}(X_G)$ given by

$$\langle \mathscr{R}[v_G] \mid \varphi \rangle = \langle \beta_* R_\nu[\omega^s]_{|s=1} \mid \varphi \rangle = \langle R_\nu[\omega^s]_{|s=1} \mid \beta^* \varphi \rangle.$$

Then $\mathscr{R}$ satisfies the Epstein-Glaser locality principle.

Renormalization group

What happens if the renormalization point ν is changed?

Theorem

Consider $(R_{\nu'} - R_\nu)[\omega^s]$ for two choices of function families ν' and ν. Locally in $U_{\underline{i}}$, applied on a test function $\varphi = \beta^*\psi$ for $\psi \in \mathcal{D}(\beta(U_{\underline{i}}) \cap \kappa_{\underline{i}}(U_{\underline{i}}))$ we have

$$\langle (R^{\underline{i}}_{\nu'} - R^{\underline{i}}_{\nu})[\omega^s_{\underline{i}}] | \varphi \rangle = \sum_{\emptyset \neq \mathcal{J} \subseteq \mathcal{N}} c_{\mathcal{J}} \langle R^{\underline{j}}_{\nu}[(\omega^s_{G//\mathcal{J}})_{\underline{j}}] | \delta_{\mathcal{J}}[\varphi] \rangle$$

with

$$c_{\mathcal{J}} := \prod_{\gamma \in \mathcal{J}} \langle R^{\underline{k}}_{\nu}[(\omega^s_{\gamma//\mathcal{J}})_{\underline{j}}] \mid \nu'_\gamma \rangle.$$

The indices $\underline{j}, \underline{k}$ correspond to $(\mathcal{N}//\mathcal{J})_{\sqsubseteq G//\mathcal{J}}$ and $(\mathcal{N}//\mathcal{J})_{\sqsubseteq \gamma//\mathcal{J}}$, respectively.

Conclusions & Outlook

- Geometric ansatz put in combinatorial language
- Simplifies the "wonderful" construction and adds discrete toolbox
- Reconstruction of Epstein-Glaser method via models for K_n $\rightarrow$ Fulton-MacPherson compactification
- Dyson-Schwinger equations?
- Renormalization group equation / flow?
- Renormalization Hopf algebra? It encodes the stratification of the exceptional divisor $\mathcal{E}$...