Discretisations of rough stochastic PDEs

Konstantin Matetski

joint work with Martin Hairer

University of Warwick

Potsdam, February 2016

The Φ^4_3 equation

Our goal: Prove that Φ^4_3 measure is invariant for Φ^4_3 equation

- Spatially periodic Φ^4_3 equation on $\mathbb{R}_+ \times \mathbb{R}^3$ is

$$\partial_t \Phi = \Delta\Phi + \infty\Phi - \Phi^3 + \xi$$

where ξ is space-time white noise, i.e. random Gaussian field,

$$\mathbb{E}\xi(t,x)\xi(s,y) = \delta_{t-s}\delta_{x-y}$$

- **Problem:** Low regularity of noise:

$$\xi \in \mathcal{C}^{-\frac{5}{2}-} \quad \Rightarrow \quad \Phi \in \mathcal{C}^{-\frac{1}{2}-} \quad \Rightarrow \quad \Phi^3 \text{ is undefined!}$$

- The bilinear map

$$\mathcal{C}^\alpha \times \mathcal{C}^\beta \ni (u,v) \mapsto uv$$

is well defined iff $\alpha + \beta > 0$

Solution of Φ^4_3 equation

Solution is defined as limit of renormalised equations

$$\partial_t \Phi_\varepsilon = \Delta \Phi_\varepsilon + C_\varepsilon \Phi_\varepsilon - \Phi_\varepsilon^3 + \xi_\varepsilon$$

- ξ_ε is a mollification of ξ so that $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$
- C_ε are renormalisation constants such that $C_\varepsilon \to \infty$

Notion of solution was provided by

- Hairer '13 (regularity structures)
- Catellier, Chouk '13 (paracontrolled distributions)
- Kupiainen '15 (renormalisation group techniques)

The Φ^4_3 measure

- Approximation of Φ^4_3 measure on dyadic lattice $\mathbb{Z}^3_\varepsilon = (\varepsilon\mathbb{Z})^3$:

$$\mu_\varepsilon(d\Phi_\varepsilon) \sim e^{-S_\varepsilon(\Phi_\varepsilon)} \prod_{x\in\mathbb{Z}^3_\varepsilon} d\Phi_\varepsilon(x)$$

- S_ε acts on functions $\Phi_\varepsilon \in \mathbb{R}^{\mathbb{Z}^3_\varepsilon}$ by

$$S_\varepsilon(\Phi_\varepsilon) = \frac{\varepsilon}{2}\sum_{x\sim y}\big(\Phi_\varepsilon(x)-\Phi_\varepsilon(y)\big)^2 - \frac{C_\varepsilon\varepsilon^3}{2}\sum_{x\in\mathbb{Z}^3_\varepsilon}\Phi_\varepsilon(x)^2 + \frac{\varepsilon^3}{4}\sum_{x\in\mathbb{Z}^3_\varepsilon}\Phi_\varepsilon(x)^4$$

- Formally $S_\varepsilon(\Phi_\varepsilon)$ is a finite difference approximation of

$$S(\Phi) = \frac{1}{2}\int_{\mathbb{R}^3}\big(\nabla\Phi(x)\big)^2\,dx - \frac{\infty}{2}\int_{\mathbb{R}^3}\Phi(x)^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3}\Phi(x)^4\,dx$$

Park '75: the Φ^4_3 measure μ is given by $\mu_\varepsilon \Rightarrow \mu$ on $\mathcal{S}'$

Our strategy of proof

Brydges, Fröhlich, Sokal '83: Moment bounds for μ_ε, implying $\mu_\varepsilon \Rightarrow \mu$ on $\mathcal{C}^{-\frac{1}{2}-}$

Our strategy of proof:

1. Consider spatial discretisations with invariant measures μ_ε:

$$\partial_t \Phi_\varepsilon = \Delta_\varepsilon \Phi_\varepsilon + C_\varepsilon \Phi_\varepsilon - \Phi_\varepsilon^3 + \xi_\varepsilon$$

2. Prove that $\Phi_\varepsilon \to \Phi$ in $\mathcal{C}([0,T], \mathcal{C}^{-\frac{1}{2}-})$ if $\Phi_\varepsilon(0) \to \Phi(0)$ in $\mathcal{C}^{-\frac{1}{2}-}$
3. Take initial conditions $\Phi_\varepsilon(0) \sim \mu_\varepsilon$ and $\Phi(0) \sim \mu$. Conclude from $\Phi_\varepsilon(0) \to \Phi(0)$ that μ is invariant for Φ^4_3 equation (argument à la Bourgain '94 for non-linear Schrödinger equation)

Remark: $\Phi_\varepsilon(0) \to \Phi(0)$ in $\mathcal{S}'$ is not sufficient.
The result by BFS is important!

Discrete Φ^4_3 equation

Consider periodic spatial discretisations of Φ^4_3 equation

$$d\Phi_\varepsilon(t,x) = \Big(\Delta_\varepsilon\Phi_\varepsilon + C_\varepsilon\Phi_\varepsilon - \Phi_\varepsilon^3\Big)(t,x)dt + \varepsilon^{-\frac{3}{2}}dW_\varepsilon(t,x)$$

on $t \in \mathbb{R}_+$ and $x \in \mathbb{Z}^3_\varepsilon$ (dyadic grid), where

- Δ_ε is nearest-neighbour discrete Laplacian

$$\Delta_\varepsilon\Phi_\varepsilon(x) = \varepsilon^{-2}\sum_{y\sim x}\big(\Phi_\varepsilon(y) - \Phi_\varepsilon(x)\big)$$

- $W_\varepsilon(\cdot,x)$, $x \in \mathbb{Z}^3_\varepsilon$, are independent (up to periodicity) Brownian motions

Fact: μ_ε is an invariant measure for this equation

Convergence of discretised Φ^4_3 equations

Theorem (Hairer, M. '15)

We assume that

- Φ is the unique maximal solution of Φ^4_3 on $[0, T^*)$
- $\|\Phi_\varepsilon(0) - \Phi(0)\|_{\mathcal{C}^\eta} \to 0$ almost surely for some $\eta > -\frac{2}{3}$

Then for every $\alpha < -\frac{1}{2}$

- there is a sequence $C_\varepsilon \sim \varepsilon^{-1}$ of renormalisation constants
- there is a sequence of stopping times T_ε satisfying $\lim_{\varepsilon \to 0} T_\varepsilon = T^*$ in probability

such that one has the limit in probability

$$\lim_{\varepsilon \to 0} \|\Phi_\varepsilon - \Phi\|_{\mathcal{C}_{\bar\eta}([0, T_\varepsilon], \mathcal{C}^\alpha)} = 0$$

for some blow-up rate $\bar\eta$ at $t = 0$

Zhu, Zhu '15: The convergence result using paracontrolled distributions

Remarks:

- Need a framework for spatial discretisations of rough SPDEs
- Want to work in spaces $\mathcal{C}([0, T], \mathcal{C}^\alpha)$

Discrete models

For a regularity structure $(\mathcal{T}, \mathcal{G})$ a discrete model $(\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ consists of

- linear maps $\Pi_x^{\varepsilon,t} : \mathcal{T} \to \mathbb{R}^{\mathbb{Z}_\varepsilon^d}$ for $t \in \mathbb{R}$, $x \in \mathbb{Z}_\varepsilon^d$
- maps $\Gamma_{xy}^{\varepsilon,t} \in \mathcal{G}$ for $t \in \mathbb{R}$ and $x, y \in \mathbb{Z}_\varepsilon^d$ such that

$$\Gamma_{xx}^{\varepsilon,t} = 1 \qquad \Gamma_{xy}^{\varepsilon,t}\,\Gamma_{yz}^{\varepsilon,t} = \Gamma_{xz}^{\varepsilon,t} \qquad \Pi_y^{\varepsilon,t} = \Pi_x^{\varepsilon,t}\,\Gamma_{xy}^{\varepsilon,t}$$

- maps $\Sigma_x^{\varepsilon,st} \in \mathcal{G}$ for $s, t \in \mathbb{R}$ and $x \in \mathbb{Z}_\varepsilon^d$ such that

$$\Sigma_x^{\varepsilon,tt} = 1 \qquad \Sigma_x^{\varepsilon,sr}\Sigma_x^{\varepsilon,rt} = \Sigma_x^{\varepsilon,st} \qquad \Sigma_x^{\varepsilon,st}\Gamma_{xy}^{\varepsilon,t} = \Gamma_{xy}^{\varepsilon,s}\Sigma_y^{\varepsilon,st}$$

For $x, y \in \mathbb{Z}_\varepsilon^d$, all test-functions φ and locally uniformly in time:

$$|\langle \Pi_x^{\varepsilon,t}\tau, \varphi_x^\lambda \rangle_\varepsilon| \lesssim \lambda^{|\tau|}\,, \qquad \lambda \in [\varepsilon, 1]$$

$$\|\Gamma_{xy}^{\varepsilon,t}\tau\|_m \lesssim |x-y|^{|\tau|-m} \qquad \|\Sigma_x^{\varepsilon,st}\tau\|_m \lesssim \left(|t-s|^{\frac{1}{2}} \vee \varepsilon\right)^{|\tau|-m}$$

Relation to original models

We can define models (Π, Γ, Σ) on $\mathbb{R}^d$ as $\varepsilon = 0$

If $(\tilde{\Pi}, \tilde{\Gamma})$ is the original model on $\mathbb{R}^{d+1}$, then

- $\tilde{\Gamma}$ is equivalent to the pair (Γ, Σ):

$$\tilde{\Gamma}_{(t,x),(s,y)} = \Gamma^t_{xy} \Sigma^{ts}_y = \Sigma^{ts}_x \Gamma^s_{xy}$$
$$\Gamma^t_{xy} = \tilde{\Gamma}_{(t,x),(t,y)} \qquad \Sigma^{st}_x = \tilde{\Gamma}_{(s,x),(t,x)}$$

- From $\tilde{\Pi}$ to Π (formally):

$$\big(\Pi^t_x \tau\big)(y) = \big(\tilde{\Pi}_{(t,x)} \tau\big)(t, y)$$

Remark: Π^t_x contains "less information" than $\tilde{\Pi}_{(t,x)}$

- From Π to $\tilde{\Pi}$:

$$\big(\tilde{\Pi}_{(t,x)} \tau\big)(s, y) = \big(\Pi^s_x \Sigma^{st}_x \tau\big)(y)$$

Discrete modeled distributions

- $H^\varepsilon : (0,T] \times \mathbb{Z}_\varepsilon^d \to \mathcal{T}$ is a modelled distribution from $\mathcal{D}_{T,\varepsilon}^{\eta,\gamma}$ if

$$\begin{aligned} \|H_t^\varepsilon(x)\|_m &\lesssim (t \vee \varepsilon^2)^{\frac{(\eta-m)\wedge 0}{2}}, \qquad m < \gamma \\ \|H_t^\varepsilon(x) - \Gamma_{xy}^{\varepsilon,t} H_t^\varepsilon(y)\|_m &\lesssim (t \vee \varepsilon^2)^{\frac{\eta-\gamma}{2}} |x-y|^{\gamma-m} \\ \|H_t^\varepsilon(x) - \Sigma_x^{\varepsilon,ts} H_s^\varepsilon(x)\|_m &\lesssim (s \vee t \vee \varepsilon^2)^{\frac{\eta-\gamma}{2}} \big(|t-s|^{\frac{1}{2}} \vee \varepsilon\big)^{\gamma-m} \end{aligned}$$

- The discrete reconstruction map

$$\big(\mathcal{R}^\varepsilon H^\varepsilon\big)_t(x) = \big(\Pi_x^{\varepsilon,t} H_t^\varepsilon(x)\big)(x) , \qquad (t,x) \in (0,T] \times \mathbb{Z}_\varepsilon^d$$

Discrete reconstruction theorem

For all $(t,x) \in (0,T] \times \mathbb{Z}_\varepsilon^d$ and all test functions φ one has

$$|\langle \big(\mathcal{R}^\varepsilon H^\varepsilon\big)_t - \Pi_x^{\varepsilon,t} H_t^\varepsilon(x), \varphi_x^\lambda \rangle_\varepsilon| \lesssim \lambda^\gamma (t \vee \varepsilon^2)^{\frac{\eta-\gamma}{2}} , \qquad \lambda \in [\varepsilon, 1]$$

Remark: The proof requires a discrete analogue of wavelets

Discrete heat kernel

- The discrete heat kernel on $(t,x) \in \mathbb{R} \times \mathbb{Z}_\varepsilon^d$:

$$G_t^\varepsilon(x) = \varepsilon^{-d} \mathbf{1}_{t\geq 0} \big(e^{t\Delta_\varepsilon} \delta_{0,\cdot}\big)(x)$$

where δ is Kronecker's delta

- Replacing $\delta_{0,\cdot}$ by a suitable smooth function we write

$$G^\varepsilon = \underbrace{K^\varepsilon}_{\text{localised}} + \underbrace{R^\varepsilon}_{\mathcal{C}^\infty}$$

- For $\varepsilon = 2^{-N}$ we expand K^ε as

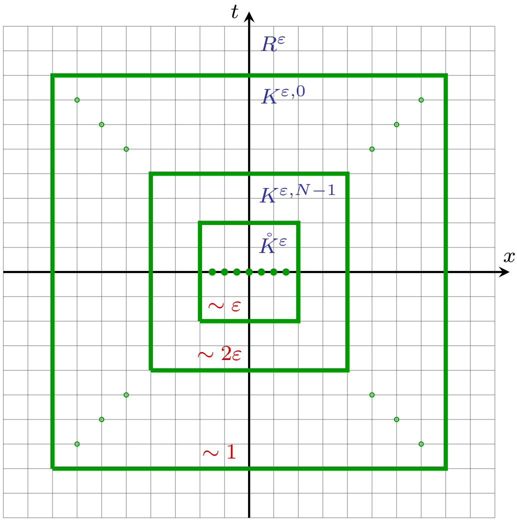
$$K^\varepsilon = \underbrace{\mathring{K}^\varepsilon}_{\text{discrete}} + \underbrace{\bar{K}^\varepsilon}_{\mathcal{C}^\infty} = \mathring{K}^\varepsilon + \sum_{n=0}^{N-1} K^{\varepsilon,n}$$

- in such a way that for some $c > 0$:

$$\operatorname{supp}(\mathring{K}^\varepsilon) \subset \operatorname{Box}(0, c\varepsilon) \qquad |\mathring{K}^\varepsilon(z)| \lesssim \varepsilon^{-d}$$

$$\operatorname{supp}(K^{\varepsilon,n}) \subset \operatorname{Box}(0, c2^{-n}) \qquad |D^k K^{\varepsilon,n}(z)| \lesssim 2^{(d+|k|)n}$$

Expansion of the kernel



Abstract integration

Let $\mathcal{I} : \mathcal{T} \to \mathcal{T}$ be an abstract integration map, i.e. $\mathcal{I} : \mathcal{T}_\alpha \to \mathcal{T}_{\alpha+2}$

Our aim: To define $\Pi^\varepsilon \mathcal{I}\tau$, $\Gamma^\varepsilon \mathcal{I}\tau$ and $\Sigma^\varepsilon \mathcal{I}\tau$ by $\Pi^\varepsilon \tau$, $\Gamma^\varepsilon \tau$, $\Sigma^\varepsilon \tau$ and K^ε

- We define

$$\big(\Pi_x^{\varepsilon,t}\mathcal{I}\tau\big)(y) = \int_{\mathbb{R}} \langle \Pi_x^{\varepsilon,s}\Sigma_x^{\varepsilon,st}\tau, K_{t-s}^\varepsilon(y-\cdot)\rangle_\varepsilon ds - \underbrace{\Pi_x^{\varepsilon,t}\left(\mathcal{J}_{t,x}^\varepsilon\tau\right)(y)}_{\text{polynomial}}$$

- where the polynomial part is given by

$$\begin{aligned} \mathcal{J}_{t,x}^\varepsilon\tau &= \mathbf{1}\int_{\mathbb{R}} \langle \Pi_x^{\varepsilon,s}\Sigma_x^{\varepsilon,st}\tau, \mathring{K}_{t-s}^\varepsilon(x-\cdot)\rangle_\varepsilon ds \\ &\quad + \sum_{|k|<|\tau|+2} \frac{X^k}{k!}\int_{\mathbb{R}} \langle \Pi_x^{\varepsilon,s}\Sigma_x^{\varepsilon,st}\tau, D^k\bar{K}_{t-s}^\varepsilon(x-\cdot)\rangle_\varepsilon ds \end{aligned}$$

- Furthermore, we assume

$$\Gamma_{xy}^{\varepsilon,t}\big(\mathcal{I}+\mathcal{J}_{t,y}^\varepsilon\big) = \big(\mathcal{I}+\mathcal{J}_{t,x}^\varepsilon\big)\Gamma_{xy}^{\varepsilon,t}\,, \qquad \Sigma_x^{\varepsilon,st}\big(\mathcal{I}+\mathcal{J}_{t,x}^\varepsilon\big) = \big(\mathcal{I}+\mathcal{J}_{s,x}^\varepsilon\big)\Sigma_x^{\varepsilon,st}$$

Abstract integration, cont.

- To get the required estimates we write

$$\big(\Pi_x^{\varepsilon,t}\mathcal{I}\tau\big)(y) = \underbrace{\int_{\mathbb{R}} \langle \Pi_x^{\varepsilon,s}\Sigma_x^{\varepsilon,st}\tau, \mathring{K}_{t-s}^{\varepsilon}(y-\cdot)\rangle_\varepsilon ds}_{=(\mathring{\Pi}_x^{\varepsilon,t}\mathcal{I}\tau)(y)} + \big(\bar{\Pi}_x^{\varepsilon,t}\mathcal{I}\tau\big)(y) + \dots$$

- $\langle \bar{\Pi}_x^{\varepsilon,t}\mathcal{I}\tau, \varphi_x^\lambda\rangle_\varepsilon$ is bounded similarly to continuous case and

$$\big|\langle \mathring{\Pi}_x^{\varepsilon,t}\mathcal{I}\tau, \varphi_x^\lambda\rangle_\varepsilon\big| \lesssim \varepsilon^{|\tau|+2} \lesssim \lambda^{|\tau|+2} \;,$$

  if $|\tau| + 2 > 0$ and $\lambda \geq \varepsilon$. Moreover,

$$\begin{aligned} \big|\langle (\mathring{\Pi}_x^{\varepsilon,t} - \mathring{\Pi}_x^{\varepsilon,s})\mathcal{I}\tau, \varphi_x^\lambda\rangle_\varepsilon\big| &\leq \big|\langle \mathring{\Pi}_x^{\varepsilon,t}\mathcal{I}\tau, \varphi_x^\lambda\rangle_\varepsilon\big| + \big|\langle \mathring{\Pi}_x^{\varepsilon,s}\mathcal{I}\tau, \varphi_x^\lambda\rangle_\varepsilon\big| \\ &\lesssim \varepsilon^{|\tau|+2} \lesssim \lambda^{|\tau|+2-\delta}\big(|t-s|^{\frac{1}{2}} \vee \varepsilon\big)^\delta \end{aligned}$$

- Somewhat similarly we can build a map $\mathcal{K}_\gamma^\varepsilon : \mathcal{D}_{T,\varepsilon}^{\eta,\gamma} \to \mathcal{D}_{T,\varepsilon}^{\eta+2,\gamma+2}$ s.t.

$$\mathcal{R}_t^\varepsilon\big(\mathcal{K}_\gamma^\varepsilon H^\varepsilon\big)_t(x) = \int_0^t \langle \mathcal{R}_s^\varepsilon H_s^\varepsilon, K_{t-s}^\varepsilon(x-\cdot)\rangle_\varepsilon ds$$

Analysis of discretised SPDEs

Consider spatial discretisations of a locally subcritical equation

$$\partial_t u_\varepsilon = \Delta_\varepsilon u_\varepsilon + F_\varepsilon(u_\varepsilon, \xi_\varepsilon)\;, \qquad u_\varepsilon(0) = u_\varepsilon^0$$

Algorithm:

1. Build a regularity structure $(\mathcal{T}, \mathcal{G})$ (as in continuous case)
2. Remove elements from $\mathcal{T}$ which do not correspond to functions in time, e.g. Ξ corresponds to ξ_ε
3. Lift canonically ξ_ε to a discrete model $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$
4. Reformulate the equation

   $$U^\varepsilon = \mathcal{K}_\gamma^\varepsilon F_\varepsilon(U^\varepsilon, \Xi) + \underbrace{\text{polynomial}}_{\text{comes from } u_\varepsilon^0 \text{ and } R^\varepsilon}$$

   where U^ε is a modeled distribution and

   $$u_\varepsilon(t,x) = \big(\mathcal{R}^\varepsilon U^\varepsilon\big)_t(x)$$

5. Show that $u_\varepsilon^0 \to u^0$ and $Z^\varepsilon \to Z$ (after renormalisation)
6. Conclude that $U^\varepsilon \to U$ in $\mathcal{D}_{T,\varepsilon}^{\eta,\gamma}$ (fixed point argument) and $u_\varepsilon \to u$

Conclusions and outlooks

Conclusions:

- The Φ^4_3 measure is invariant for the Φ^4_3 equation
- For almost every (wrt Φ^4_3 measure) initial condition, the solution to Φ^4_3 equation is almost surely global in time
- The framework can be applied to many rough SPDEs, incl. KPZ, Burgers, parabolic Anderson etc.

Question:

- What about non-Gaussian and non-stationary noise, e.g. martingale-driven equations coming from particle systems?