Elementary Renormalisation in linear algebra and dynamical systems

Frédéric Menous

Université Paris-Sud

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Perturbative expansion and Lie algebra.

Let $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_\nu)$, $V \in M_\nu(\mathbb{C})$ with $\alpha V \in L = M_\nu(\mathbb{C}_{\geqslant 1}[[\alpha]])$. If one can solve

$$(\Lambda + \alpha V)P = P\Lambda$$

with $P = I_\nu + \alpha P^{[1]} + \alpha^2 P^{[2]} + \ldots \in G = \exp(L) = I_\nu + M_\nu(\mathbb{C}_{\geqslant 1}[[\alpha]])$.

→ The above equation relates $\alpha V \in L$ (Lie algebra) to $P \in G = \exp(L)$ (Lie group).

→ The matrices $\Lambda + \alpha V$ and Λ should be similar :

(ALMOST) NO HOPE

→ Perturbated matrix does not mean similar matrix.

→ Even if it doesn't work, can we get some information on the perturbated system ?

Perturbative theories and renormalization.

	pQFT	pLinear Algebra
Compute	Feynman Integrals	Change of basis
Structure	Group of characters of a Hopf algebra	Group $I_\nu + M_\nu(\mathbb{C}_{\geqslant 1}[[\alpha]])$
Difficulty	Divergence in some dimension	Non similarity
Regularization	DimReg	???
Use the group	Birkhoff decomposition	???
Get the right result	$\varepsilon = 0$	???

Remarks : In the last

- The perturbation leaves in a completed graded Lie Algebra

$$\alpha V \in L = M_\nu(\mathbb{C}_{\geqslant 1}[[\alpha]]) = L_1 + L_2 + \ldots + L_n + \ldots.$$

- The group we deal with is its Lie group $G = \exp(L)$

- G is also a group of character on a commutative algebra

- Once the linear par Λ is fixed, there is a graded derivation d_Λ on L such that the equation that relates $\alpha V \in L$ to $P \in G$ is a logarithmic derivative

$$(d_\Lambda P)P^{-1} = \alpha V$$

- The following results hold in this framework (FM, 2013)

- This simple case seems to be related to Rayleigh-Schrödinger perturbative theory (C. Brouder et al.).....

Back to linear algebra

Let $\alpha V \in L = M_\nu(\mathbb{C}_{\geqslant 1}[[\alpha]])$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_\nu)$, $P = I_\nu + \alpha P^{[1]} + \alpha^2 P^{[2]} + \ldots$:

$$(\Lambda + \alpha V)P = P\Lambda \Longleftrightarrow d_\Lambda(P) = [P, \Lambda] = \alpha V P$$

The linear map d_Λ is a graded derivation since Λ is of degree 0 in α and

$$d_\Lambda[A, B] = [d_\Lambda(A), B] + [A, d_\Lambda(B)] \text{ (Jacobi)}$$

Perturbatively ($P^{[0]} = I_\nu$):

$$d_\Lambda\big(P^{[1]}\big) = V, d_\Lambda\big(P^{[2]}\big) = V P^{[1]}, \ldots, d_\Lambda\big(P^{[n]}\big) = V P^{[n-1]}, \ldots.$$

But d_Λ is not invertible. The Sylvester equation $d_\Lambda(A) = [A, \Lambda] = B$ reads:

$$[A, \Lambda]_{i,j} = (\lambda_j - \lambda_i) A_{i,j} = B_{i,j}$$

In the canonical basis $\{E^{i,j}\}$ of the $\mathbb{C}_{\geqslant 1}[[\alpha]]$–module L:

$$\mathrm{Ker}\, d_\Lambda = \mathrm{span}\{E^{i,j}\,;\, \lambda_i = \lambda_j\}, \mathrm{Im}\, d_\Lambda = \mathrm{span}\{E^{i,j}\,;\, \lambda_i \neq \lambda_j\}$$

$$L = \mathrm{Ker}\, d_\Lambda \oplus \mathrm{Im}\, d_\Lambda.$$

Let's regularize ...

There is a natural graded invertible derivation on $L = M_\nu(\mathbb{C}_{\geqslant 1}[[\alpha]])$: $\alpha\partial_\alpha$ (Y).

Regularization :

$$d_\Lambda(P) = [P, \Lambda] = \alpha VP \longrightarrow (d_\Lambda + \varepsilon\alpha\partial_\alpha)P = \alpha VP$$

Perturbatively $P = I_\nu + \alpha P^{[1]} + \alpha^2 P^{[2]} + \ldots$ $(P^{[0]} = I_\nu)$:

$$\left[P^{[1]}, \Lambda\right] + \varepsilon P^{[1]} = V \quad : \quad (\lambda_j - \lambda_i + \varepsilon)P^{[1]}_{i,j} = V_{i,j} \quad : \quad P^{[1]}_{i,j} = -\frac{V_{i,j}}{\lambda_i - \lambda_j - \varepsilon}$$

$$\left[P^{[2]}, \Lambda\right] + 2\varepsilon P^{[2]} = VP^{[1]} \quad : \quad (\lambda_j - \lambda_i + 2\varepsilon)P^{[2]}_{i,j} = \left(VP^{[1]}\right)_{i,j} \quad : \quad P^{[2]}_{i,j} = \sum_k \frac{V_{i,k}V_{k,j}}{(\lambda_i - \lambda_j - 2\varepsilon)(\lambda_k - \lambda_j - \varepsilon)}$$

$$P^{[n]}_{i,j} = (-1)^n \sum_{k_1,\ldots,k_{n-1}} \frac{V_{i,k_{n-1}}V_{k_{n-1},k_{n-2}}\ldots V_{k_1,j}}{(\lambda_i - \lambda_j - n\varepsilon)(\lambda_{k_{n-1}} - \lambda_j - (n-1)\varepsilon)\ldots(\lambda_{k_1} - \lambda_j - \varepsilon)}$$

The regularized equation has a solution P_ε in $I_\nu + M_\nu(\alpha\mathcal{A}[[\alpha]])$ where

$\mathcal{A} = \mathbb{C}[[\varepsilon]][\varepsilon^{-1}]$ is the algebra of Laurent series.

… and renormalize

1. There exists a unique factorization $P_\varepsilon = P_\varepsilon^+ P_\varepsilon^-$ with

$$P_\varepsilon^+ \in I_\nu + M_\nu(\alpha\mathbb{C}[[\varepsilon,\alpha]])\,,\, P_\varepsilon^- \in I_\nu + M_\nu(\varepsilon^{-1}\alpha\mathbb{C}[\varepsilon^{-1}][[\alpha]])$$

   this is the Birkhoff decomposition.

2. There exists $N \in M_\nu(\alpha\mathbb{C}[[\alpha]])$ independant of ε, commuting with Λ, such that

$$P_0^+(\Lambda + N) = (\Lambda + \alpha V)P_0^+$$

3. In fact P_ε^- commutes with Λ and

$$\varepsilon\alpha\partial_\alpha P_\varepsilon^- = N P_\varepsilon^-$$

If all the λ_i are distinct, $\Lambda + N$ is diagonal and similar to $\Lambda + \alpha V$. This totally determines the eigenvalues of $\Lambda + \alpha V$ and its eigenvectors (columns of P_0^+).

If some λ_i are equal, then $\Lambda + N$ is block diagonal with block dimensions corresponding to the multiplicities of the initial eigenvalues λ_i. This splits the the vector space into a direct sum of subspaces that are stable by $\Lambda + \alpha V$.

(1) Birkhoff decomposition.

We have $P_\varepsilon = I_\nu + \alpha P_\varepsilon^1 + \alpha^2 P_\varepsilon^2 + \ldots.$ with P_ε^1, P_ε^2, ...with coefficients in Laurent series. If $P_\varepsilon = P_\varepsilon^+ P_\varepsilon^-$ with $P_\varepsilon^\pm = I_\nu + \alpha P_\varepsilon^{1,\pm} + \alpha^2 P_\varepsilon^{2,\pm} + \ldots$ then

$$\underbrace{P_\varepsilon^{1,+}}_{\text{regular}} + \underbrace{P_\varepsilon^{1,-}}_{\text{polar}} = \underbrace{P_\varepsilon^{1}}_{\text{reg+polar}}$$

$$\underbrace{P_\varepsilon^{2,+}}_{\text{regular}} + \underbrace{P_\varepsilon^{1,+}P_\varepsilon^{1,-}}_{\text{reg+polar}} + \underbrace{P_\varepsilon^{2,-}}_{\text{polar}} = \underbrace{P_\varepsilon^{2}}_{\text{reg+polar}}$$

And so on ....

Plug into the equation ($d_\Lambda = [\,.\,,\Lambda]$)

$$(d_\Lambda + \varepsilon\alpha\partial_\alpha)P_\varepsilon = \alpha V P_\varepsilon$$

$$[(d_\Lambda + \varepsilon\alpha\partial_\alpha)P_\varepsilon^+]P_\varepsilon^- + P_\varepsilon^+[(d_\Lambda + \varepsilon\alpha\partial_\alpha)P_\varepsilon^-] = \alpha V P_\varepsilon^+ P_\varepsilon^-$$

$$\underbrace{(P_\varepsilon^+)^{-1}[(d_\Lambda + \varepsilon\alpha\partial_\alpha)P_\varepsilon^+]}_{\text{regular}} + \underbrace{[(d_\Lambda + \varepsilon\alpha\partial_\alpha)P_\varepsilon^-](P_\varepsilon^-)^{-1}}_{\text{polar+constant}} = \underbrace{\alpha(P_\varepsilon^+)^{-1} V P_\varepsilon^+}_{\text{regular}}$$

Thus $N = [(d_\Lambda + \varepsilon\alpha\partial_\alpha)P_\varepsilon^-](P_\varepsilon^-)^{-1}$ does not depend on ε and, for $\varepsilon = 0$, we get $P_0^+(\Lambda + N) = (\Lambda + \alpha V)P_0^+$.

Summary

$$(d_\Lambda + \varepsilon\alpha\partial_\alpha)P_\varepsilon = \alpha V P_\varepsilon, \quad (d_\Lambda = [\,.\,,\Lambda])$$

1. There exists a unique factorization $P_\varepsilon = P_\varepsilon^+ P_\varepsilon^-$ with

$$P_\varepsilon^+ \in I_\nu + M_\nu(\alpha\mathbb{C}[[\varepsilon,\alpha]])\,, P_\varepsilon^- \in I_\nu + M_\nu(\varepsilon^{-1}\alpha\mathbb{C}[\varepsilon^{-1}][[\alpha]])$$

2. We have

$$P_0^+(\Lambda + N) = (\Lambda + \alpha V)P_0^+$$

where $N = [(d_\Lambda + \varepsilon\alpha\partial_\alpha)P_\varepsilon^-](P_\varepsilon^-)^{-1}$ independent of ε.

3. If $[P_\varepsilon^-, \Lambda] = 0$, $N = [(d_\Lambda + \varepsilon\alpha\partial_\alpha)P_\varepsilon^-](P_\varepsilon^-)^{-1} = [\varepsilon\alpha\partial_\alpha P_\varepsilon^-](P_\varepsilon^-)^{-1}$: $N \in M_\nu(\alpha\mathbb{C}[[\alpha]])$ independant of ε and

$$\varepsilon\alpha\partial_\alpha P_\varepsilon^- = N P_\varepsilon^-, \quad \text{with}\,[N,\Lambda] = 0$$

(2)-(3) Commutation with Λ

It remains to prove that $d_\Lambda(P_\varepsilon^-) = 0$ and then, necessarily:

$$0 = \varepsilon\alpha\partial_\alpha(d_\Lambda(P_\varepsilon^-)) = d_\Lambda(\varepsilon\alpha\partial_\alpha P_\varepsilon^-) = d_\Lambda(NP_\varepsilon^-) = d_\Lambda(N).P_\varepsilon^-$$

thus $[N, \Lambda] = 0$.

If $N = N^0 + \alpha N^1 + \alpha^2 N^2 + \ldots.$, then $N^0 = 0$ and, since $(d_\Lambda + \varepsilon\alpha\partial_\alpha)P_\varepsilon^- = NP_\varepsilon^-$,

$$d_\Lambda\left(P_\varepsilon^{1,-}\right) + \varepsilon P_\varepsilon^{1,-} = N^1$$

If $P_\varepsilon^{1,-} = \sum_{1 \leqslant k \leqslant N} C_k \varepsilon^{-k}$ with $C_N \neq 0$,

$$\sum_{1 \leqslant k \leqslant N} d_\Lambda(C_k)\varepsilon^{-k} + \sum_{0 \leqslant k \leqslant N-1} C_{k+1}\varepsilon^{-k} = N^1$$

Thus $d_\Lambda(C_N) = 0$ ($C_N \in \operatorname{Ker} d_\Lambda$). If $N \geqslant 2$, $d_\Lambda(C_{N-1}) + C_N = 0$ thus $C_N \in \operatorname{Im} d_\Lambda$ and $C_N = 0$ (Impossible). If $N = 1$, $C_1 = N^1$ and $d_\Lambda(C_1) = d_\Lambda(N^1) = d_\Lambda\left(P_\varepsilon^{1,-}\right) = 0$.

And so on ....

Summary

$$(d_\Lambda + \varepsilon\alpha\partial_\alpha)P_\varepsilon = \alpha V P_\varepsilon, \quad (d_\Lambda = [\,.\,, \Lambda])$$

1. There exists a unique factorization $P_\varepsilon = P_\varepsilon^+ P_\varepsilon^-$ with

$$P_\varepsilon^+ \in I_\nu + M_\nu(\alpha\mathbb{C}[[\varepsilon, \alpha]])\,, P_\varepsilon^- \in I_\nu + M_\nu(\varepsilon^{-1}\alpha\mathbb{C}[\varepsilon^{-1}][[\alpha]])$$

2. $[P_\varepsilon^-, \Lambda] = 0$ and there exists $N \in M_\nu(\alpha\mathbb{C}[[\alpha]])$ independant of ε such that

$$\varepsilon\alpha\partial_\alpha P_\varepsilon^- = N P_\varepsilon^-, \quad [N, \Lambda] = 0$$

3. We have

$$P_0^+(\Lambda + N) = (\Lambda + \alpha V)P_0^+$$

Residue and Dynkin map

From the equation $\varepsilon\alpha\partial_\alpha P_\varepsilon^- = N P_\varepsilon^-$ we get :

$$\alpha\partial_\alpha \mathrm{Res}(P_\varepsilon^-) = N \quad \text{(Beta function)}$$

If $N = \alpha N^1 + \alpha^2 N^2 + \ldots$, since $\alpha\partial_\alpha$ is the grading operator :

$$P_\varepsilon^- = I_\nu + \sum_{\substack{s\geqslant 1 \\ k_1,\ldots,k_s\geqslant 1}} \alpha^{k_1+\ldots+k_s} \frac{N^{k_s}\ldots N^{k_1}}{\varepsilon^s(k_1+\ldots+k_s)(k_1+\ldots+k_{s-1})\ldots k_1}$$

This is the Dynkin map.

Locality, differential equation and adiabatic limit.

$$(d_\Lambda + \varepsilon\alpha\partial_\alpha)P_\varepsilon = \alpha V P_\varepsilon,\, P_\varepsilon = I_\nu + \alpha P_\varepsilon^1 + \alpha^2 P_\varepsilon^2 + \dots. \quad (d_\Lambda = [\,.\,,\Lambda])$$

Consider the one-parameter family $P_\varepsilon(t) = I_\nu + \alpha e^{\varepsilon t} P_\varepsilon^1 + \alpha^2 e^{2\varepsilon t} P_\varepsilon^2 + \dots$

1. If $P_\varepsilon(t) = P_\varepsilon(t)^+ P_\varepsilon(t)^-$ then $P_\varepsilon(t)^- = P_\varepsilon(0)^- = P_\varepsilon^-$: “Locality”
2. Consider $U_\varepsilon(t) = P_\varepsilon(t) e^{\Lambda t}$ then $\partial_t U_\varepsilon(t) = (\Lambda + \alpha e^{\varepsilon t} V) U_\varepsilon(t)$ with

$$U_\varepsilon(t)_{i,j} = \delta_{i,j} + \sum_{\substack{n \geqslant 1 \\ k_1, \dots, k_{n-1}}} \frac{(-1)^n \alpha^n e^{n\varepsilon t} e^{\lambda_j t} V_{i,k_{n-1}} V_{k_{n-1},k_{n-2}} \dots V_{k_1,j}}{(\lambda_i - \lambda_j - n\varepsilon)(\lambda_{k_{n-1}} - \lambda_j - (n-1)\varepsilon) \dots (\lambda_{k_1} - \lambda_j - \varepsilon)}$$

   and “$\lim_{\varepsilon t \to -\infty} U_\varepsilon(t) = I_\nu$” but no adiabatic limit $t = 0$, $\varepsilon = 0$.

3. $U_\varepsilon^+(t) = U_\varepsilon(t)(P_\varepsilon^-)^{-1} = P_\varepsilon(t) e^{\Lambda t} (P_\varepsilon^-)^{-1} = P_\varepsilon(t)^+ e^{\Lambda t}$, then

$$\partial_t U_\varepsilon^+(t) = (\Lambda + \alpha e^{\varepsilon t} V) U_\varepsilon^+(t)$$

   and $\lim_{t=0,\varepsilon=0} U_\varepsilon^+(t) = P_0^+$ such that $P_0^+(\Lambda + N) = (\Lambda + \alpha V) P_0^+$.

We have

$$(d_\Lambda + \partial_t) P_\varepsilon(t) = \alpha e^{\varepsilon t} V . P_\varepsilon(t)$$

This give $(d_\Lambda + \partial_t) P_\varepsilon(t)^- = 0$ and if $P_\varepsilon(t)^- = P_\varepsilon^- + \sum_{k\geqslant 1} t^k C_k$ then $C_1 + d_\Lambda(P_\varepsilon^-) = 0$ thus $C_1 = 0$ and so on....

Dynamical sytems : perturbative vs conjugacy

$$\left\{\begin{array}{lcl} \frac{dx_1}{dt} & = & \lambda_1 x_1 \\ \vdots & & \vdots \\ \frac{dx_\nu}{dt} & = & \lambda_\nu x_\nu \end{array}\right. \overset{???}{\longleftrightarrow} \left\{\begin{array}{lcl} \frac{dx_1}{dt} & = & \lambda_1 x_1 + V_1(x_1, \ldots, x_\nu) \\ \vdots & & \vdots \\ \frac{dx_\nu}{dt} & = & \lambda_\nu x_\nu + V_\nu(x_1, \ldots, x_\nu) \end{array}\right.$$

Nonlinear perturbation and linearization

Let $\boldsymbol{x} = (x_1, \ldots, x_\nu)$, $\Lambda.\boldsymbol{x} = (\lambda_1 x_1, \ldots, \lambda_\nu x_\nu)$ and $V = (V_1, \ldots, V_\nu) \in \mathbb{C}_{\geqslant 2}\{\boldsymbol{x}\}^\nu$ (analytic or even formal) :

$$\frac{d\boldsymbol{x}}{dt} = \Lambda.\boldsymbol{x} + V(\boldsymbol{x}) \quad \underset{\boldsymbol{x}=\varphi^{-1}(\boldsymbol{y})}{\overset{\boldsymbol{y}=\varphi(\boldsymbol{x})}{\longleftrightarrow}} \quad \frac{d\boldsymbol{y}}{dt} = \Lambda.\boldsymbol{y}$$

with $\varphi(x_1, \ldots, x_\nu) = (x_1 + h.o.t, \ldots, x_\nu + h.o.t) \in \mathbb{C}\{\boldsymbol{x}\}^\nu$ analytic (or formal) **identity-tangent diffeomorphism** (group).

The homological equation:

$$\frac{d\boldsymbol{y}}{dt} = \frac{d\varphi(\boldsymbol{x})}{dt} = \sum_{i=1}^{\nu} \frac{dx_i}{dt}\frac{\partial \varphi}{\partial x_i}(\boldsymbol{x}) = \sum_{i=1}^{\nu} (\lambda_i x_i + V_i(\boldsymbol{x}))\frac{\partial \varphi}{\partial x_i}(\boldsymbol{x}) = \Lambda.\varphi(\boldsymbol{x}) = \Lambda.\boldsymbol{y}$$

relates $V \in L$ (Lie algebra) to $\varphi \in G = \exp(L)$ (Lie group).

Poincaré, Birkhoff, ..., Brjuno, ...:

→ Nonresonant case: formal solution. Analyticity under diophantine condition.

→ Resonant case: no formal solution: Perturbated systems differ from conjugate system.

Linear perturbation.

If $V(\boldsymbol{x}) = \alpha V.\boldsymbol{x}$ with $\alpha V \in L = M_\nu(\mathbb{C}_{\geqslant 1}[[\alpha]])$.

$$\frac{d\boldsymbol{x}}{dt} = (\Lambda + \alpha V).\boldsymbol{x} + \underset{\boldsymbol{x}=P^{-1}.\boldsymbol{y}}{\overset{\boldsymbol{y}=P.\boldsymbol{x}}{\longleftrightarrow}} \frac{d\boldsymbol{y}}{dt} = \Lambda.\boldsymbol{y}$$

with $P = I_\nu + \alpha P^1 + \alpha^2 P^2 + \ldots \in G = \exp(L) = I_\nu + M_\nu(\mathbb{C}_{\geqslant 1}[[\alpha]])$.

→ The corresponding equation :

$$(\Lambda + \alpha V)P = P\Lambda$$

relates $\alpha V \in L$ (Lie algebra) to $P \in G = \exp(L)$ (Lie group).

→ The matrices $\Lambda + \alpha V$ and Λ should be similar :

(ALMOST) NO HOPE

→ Perturbated matrix does not mean similar matrix.

→ Even if it doesn't work, can we get some information on the perturbated system ?

Perturbative theories an renormalization.

	pQFT	Dynamical systems
Compute	Feynman Integrals	Coefficients of a diffeomorphism
Structure	Group of characters on a Hopf algebra	Group of diffeomorphisms
Difficulty	Divergence in some dimension	Resonant vector fields
Regularization	DimReg	???
Use the group	Birkhoff decomposition	???
Get the right result	$\varepsilon = 0$	???

Back to conjugacy of formal vector fields.

In dim. ν : $(\lambda_1, \ldots, \lambda_\nu) \in \mathbb{C}^\nu$, $a = (a_1, \ldots, a_\nu), b = (b_1, \ldots b_\nu) \in (\mathbb{C}_{\geqslant 2}[[x_1, \ldots, x_\nu]])^\nu$:

$$\begin{array}{ccccc} \frac{dx_1}{dt} & = & A_1(\boldsymbol{x}) & = & \lambda_1 x_1 + a_1(\boldsymbol{x}) \\ \vdots & & & & \vdots \\ \frac{dx_\nu}{dt} & = & A_\nu(\boldsymbol{x}) & = & \lambda_\nu x_\nu + a_\nu(\boldsymbol{x}) \end{array} \quad \overset{\boldsymbol{y} = \varphi(\boldsymbol{x})}{\longleftrightarrow} \quad \begin{array}{ccccc} \frac{dy_1}{dt} & = & B_1(\boldsymbol{y}) & = & \lambda_1 y_1 + b_1(\boldsymbol{y}) \\ \vdots & & & & \vdots \\ \frac{dy_\nu}{dt} & = & B_\nu(\boldsymbol{y}) & = & \lambda_\nu y_\nu + b_\nu(\boldsymbol{y}) \end{array}$$

where $\varphi = (\varphi_1, \ldots, \varphi_\nu) = (x_1 + u_1, \ldots x_\nu + u_\nu)$, $u = (u_1, \ldots, u_\nu) \in (\mathbb{C}_{\geqslant 2}[[x_1, \ldots, x_\nu]])^\nu$.

The conjugacy equation reads : $\forall\ 1 \leqslant j \leqslant \nu$

$$\frac{dy_j}{dt} = \sum_{i=1}^{\nu} \frac{dx_i}{dt} \frac{\partial \varphi_j}{\partial x_i} = \sum_{i=1}^{\nu} A_i(\boldsymbol{x}) \partial_{x_i} \varphi_j = B_j \circ \varphi(\boldsymbol{x}) = B_j(\boldsymbol{y})$$

On $f \in \mathbb{C}[[\boldsymbol{x}]]$: $\frac{d}{dt} f(\boldsymbol{x}) = \sum_{i=1}^{\nu} A_i(\boldsymbol{x}) \partial_{x_i} f(\boldsymbol{x}) = X_A.f$, $f \circ \varphi(x) = (F_\varphi.f)(\boldsymbol{x})$:

$$\forall 1 \leqslant j \leqslant \nu, \quad X_A.F_\varphi.x_j = F_\varphi.X_B.x_j$$

$$X_A.F_\varphi = F_\varphi.X_B$$

Vector fields, derivations and homogeneous degrees

Let $\boldsymbol{n}=(n_1,\ldots,n_\nu)\in\mathbb{N}^\nu$, $|\boldsymbol{n}|=n_1+\ldots+n_\nu$, $\boldsymbol{x}^{\boldsymbol{n}}=x_1^{n_1}\ldots x_\nu^{n_\nu}$: $\mathrm{td}(\boldsymbol{x}^{\boldsymbol{n}})=|\boldsymbol{n}|$.

$$X_A=\sum_{i=1}^{\nu}\lambda_i x_i\partial_{x_i}+\sum_{i=1}^{\nu}a_i(\boldsymbol{x})\partial_{x_i}=\sum_{i=1}^{\nu}\lambda_i x_i\partial_{x_i}+\sum_{i=1}^{\nu}\sum_{|\boldsymbol{n}|\geqslant 2}a_i^{\boldsymbol{n}}\boldsymbol{x}^{\boldsymbol{n}}\partial_{x_i}$$

If $\boldsymbol{n}=(n_1,\ldots,n_\nu)$ and $\boldsymbol{m}=(m_1,\ldots,m_\nu)$:

$$\lambda_i x_i\partial_{x_i}.\boldsymbol{x}^{\boldsymbol{m}}=m_i\lambda_i\boldsymbol{x}^{\boldsymbol{m}}\quad,\quad a_i^{\boldsymbol{n}}\boldsymbol{x}^{\boldsymbol{n}}\partial_{x_i}.\boldsymbol{x}^{\boldsymbol{m}}=a_i^{\boldsymbol{n}}x_1^{n_1+m_1}\ldots x_i^{n_i+m_i-1}\ldots x_\nu^{n_\nu+m_\nu}$$

Thus

$$\mathrm{td}(\lambda_i x_i\partial_{x_i}.\boldsymbol{x}^{\boldsymbol{m}})=0+\mathrm{td}(\boldsymbol{x}^{\boldsymbol{m}})\quad,\quad \mathrm{td}(a_i^{\boldsymbol{n}}\boldsymbol{x}^{\boldsymbol{n}}\partial_{x_i}.\boldsymbol{x}^{\boldsymbol{m}})=|\boldsymbol{n}|-1+\mathrm{td}(\boldsymbol{x}^{\boldsymbol{m}}).$$

→ Homogenous degree : $\mathrm{hd}(\lambda_i x_i\partial_{x_i})=0$, $\mathrm{hd}(a_i^{\boldsymbol{n}}\boldsymbol{x}^{\boldsymbol{n}}\partial_{x_i})=|\boldsymbol{n}|-1$

→ The operator X_A decomposes in homogeneous components :

$$X_A=X_A^0+\sum_{k\geqslant 1}X_A^k$$

→ Note that if X,Y are homogeneous derivations, then $[X,Y]=X.Y-Y.X$ is a derivation and $\mathrm{hd}([X,Y])=\mathrm{hd}(X)+\mathrm{hd}(Y)$....

Diffeos, operators and homogeneous degrees

In dimension ν, with coordinates $\boldsymbol{x}=(x_1,\ldots,x_\nu)$, let $\varphi(\boldsymbol{x})=(\varphi_1(\boldsymbol{x}),\ldots,\varphi_\nu(\boldsymbol{x}))$

$$\varphi_i(x)=x_i+\sum_{|\boldsymbol{n}|\geq 2}\varphi^i_{\boldsymbol{n}}\boldsymbol{x}^{\boldsymbol{n}}=x_i+u_i(\boldsymbol{x})=x_i+\sum_{|\boldsymbol{n}|\geqslant 2}u_i^{\boldsymbol{n}}\boldsymbol{x}^{\boldsymbol{n}}\quad u_i(\boldsymbol{x})\in\mathbb{C}_{\geqslant 2}[[x_1,\ldots,x_\nu]]$$

a formal identity–tangent diffeomorphism and $F_\varphi.f(\boldsymbol{x})=f\circ\varphi(\boldsymbol{x})=f(\boldsymbol{x}+u(\boldsymbol{x}))$.
The Taylor expansion of $f(\boldsymbol{x}+u(\boldsymbol{x}))$ gives:

$$F_\varphi.f(\boldsymbol{x})=f(\boldsymbol{x}+u(\boldsymbol{x}))=f(\boldsymbol{x})+\sum_{\substack{s\geqslant 1\\ 1\leqslant i_1,\ldots,i_s\leqslant\nu}}\frac{1}{s!}u_{i_1}\ldots u_{i_s}\partial_{x_{i_1}}\ldots\partial_{x_{i_s}}f(\boldsymbol{x})$$

$$F_\varphi.f(\boldsymbol{x})=\left(\mathrm{Id}+\sum_{\substack{s\geqslant 1\\ 1\leqslant i_1,\ldots,i_s\leqslant\nu\\ 2\leqslant|\boldsymbol{n}^1|,\ldots,|\boldsymbol{n}^s|}}\frac{1}{s!}u_{i_1}^{\boldsymbol{n}^1}\ldots u_{i_s}^{\boldsymbol{n}^s}\boldsymbol{x}^{\boldsymbol{n}^1+\ldots+\boldsymbol{n}^s}\partial_{x_{i_1}}\ldots\partial_{x_{i_s}}\right)f(\boldsymbol{x})$$

and

$$\mathrm{td}\left(\frac{1}{s!}u_{i_1}^{\boldsymbol{n}^1}\ldots u_{i_s}^{\boldsymbol{n}^s}\boldsymbol{x}^{\boldsymbol{n}^1+\ldots+\boldsymbol{n}^s}\partial_{x_{i_1}}\ldots\partial_{x_{i_s}}\boldsymbol{x}^{\boldsymbol{m}}\right)=|\boldsymbol{n}^1|+\ldots+|\boldsymbol{n}^s|-s+\mathrm{td}(\boldsymbol{x}^{\boldsymbol{m}})$$

thus

$$F_\varphi=\mathrm{Id}+\sum_{k\geq 1}F_\varphi^k\quad\mathrm{hd}(F_\varphi^k)=k$$

1. If $L=\{X=\sum_{n\geqslant 1} X_n, X \text{ derivation}\}$, $G=\{F_\varphi=\mathrm{Id}+\sum_{n\geqslant 1} F_n\}$ is the group of substitutions automorphisms :

$$X \in L \quad \xrightarrow{\exp} \quad F=\exp(X)=\sum \frac{X^s}{s!} \in G$$

$$X=\log(\mathrm{Id}+\tilde{F})=\sum \frac{(-1)^{s-1}\tilde{F}^s}{s} \in L \quad \xleftarrow{\log} \quad F=\mathrm{Id}+\tilde{F} \in G$$

2. Conjugacy with $X_A=X_0+\tilde{X}_A$, $X_B=X_0+\tilde{X}_B$, $F=F_\varphi$:

$$(X_0+\tilde{X}_A)F=X_A.F_\varphi=F_\varphi.X_B=F(X_0+\tilde{X}_B)$$

thus

$$\mathrm{ad}_{X_0}(F)=[X_0,F]=F\tilde{X}_B-\tilde{X}_A F$$

3. Linearization : if $A(\boldsymbol{x})=(\lambda_1 x_1,\ \ldots,\ \lambda_\nu x_\nu)$, then $X_A=X_0$ and X_B is linearizable if there exists F such that

$$\mathrm{ad}_{X_0}(F)=[X_0,F]=F\tilde{X}_B$$

4. Jacobi identity or $d=\mathrm{ad}_{X_0}$, $X,Y\in L$:

$$d([X,Y])=[d(X),Y]+[X,d(Y)]$$

and if $\mathrm{hd}(X^k)=k$, then $\mathrm{hd}(d(X^k))=k$ (preserves the “graduation”)