Asymptotic expansions and Dyson-Schwinger equations

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We will analyze a class of power series $F_\beta^\alpha \subset \mathbb{R}[[x]]$ with $\alpha, \beta > 0$,

$$f(x) = \sum_{n=0}^{\infty} f_n x^n \in F_\beta^\alpha$$

with coefficients which satisfy,

$$\lim_{n\to\infty} \frac{f_n}{\alpha^n \Gamma(n+\beta)} = C$$

- and $\tilde{f}_n = f_n - C\alpha^n\Gamma(n+\beta)$

$$\sum_{n=0}^{\infty} \tilde{f}_{n+1} x^n \in F_\beta^\alpha.$$

- These are the power series which admit an asymptotic expansion of the form,

$$f_n = \alpha^{n+\beta}\Gamma(n+\beta)\left(c_0 + \frac{c_1}{n} + \frac{c_2}{n(n-1)} + \dots\right)$$

  including power series with $\lim_{n\to\infty} \frac{f_n}{\alpha^n\Gamma(n+\beta)} = 0 \Rightarrow c_k = 0$ for all $k \geq 0$.
- These power series appear in
  - Graph and permutation counting problems in combinatorics.
  - Perturbation expansions in physics.
- Subclass of *gevrey-1*-power series.

- Consider a power series $f(x) \in F_\beta^\alpha$:

$$f_n = \alpha^{n+\beta}\Gamma(n+\beta)\left(c_0 + \frac{c_1}{n} + \frac{c_2}{n(n-1)} + \dots\right)$$

- Idea: Interpret the coefficients c_k of the **asymptotic** expansion as a new power series.

Definition

$\mathcal{A}$ maps a power series to its asymptotic expansion:

$$\begin{aligned} \mathcal{A} \quad : \quad & F_\beta^\alpha && \rightarrow && \mathbb{R}[[x]] \\ & f(x) && \mapsto && \gamma(x) = \sum_{k=0}^{\infty} c_k x^k \end{aligned}$$

Theorem 1

$\mathcal{A}$ is a derivation on F_β^α:

$$(\mathcal{A}f(x)g(x))(x) = f(x)(\mathcal{A}g)(x) + (\mathcal{A}f)(x)g(x)$$

- Follows from the *log-convexity* of Γ.

$\Rightarrow$ F_β^α is a subring of $\mathbb{R}[[x]]$.

Proof sketch

With $h(x) = f(x)g(x)$,

$$h_n = \underbrace{\sum_{k=0}^{R-1} f_{n-k} g_k + \sum_{k=0}^{R-1} f_k g_{n-k}}_{\text{High order times low order}} + \underbrace{\sum_{k=R}^{n-R} f_k g_{n-k}}_{\mathcal{O}(\alpha^n \Gamma(n+\beta-R))}$$

.

Derivative ∂

- Analyze ∂, the ordinary derivative on power series,

$$\begin{aligned} \partial \quad : \quad & \mathcal{F}^{\alpha}_{\beta} && \to && \mathcal{F}^{\alpha}_{\beta+2}, \\ & f(x) && \mapsto && f'(x) = \sum_{n=1}^{\infty} n f_n x^{n-1} \end{aligned}$$

- where the $\beta + 2$ comes from $(n+1) f_{n+1} \sim \Gamma(n + \beta + 2)$.

- We have the commutative diagram,

$$\begin{array}{ccc} F_\beta^\alpha & \xrightarrow{\partial} & F_{\beta+2}^\alpha \\ \big\downarrow \mathcal{A} & & \big\downarrow \mathcal{A} \\ \mathbb{R}[[x]] & \xrightarrow{\partial^\mathcal{A}} & \mathbb{R}[[x]] \end{array}$$

  with $\partial^\mathcal{A} = \alpha^{-1} - x\beta + x^2\partial$
- where $\partial^\mathcal{A}$ is a bijection, because $\ker \partial \subset \ker \mathcal{A}$!

- What happens for composition of power series $\in F_\beta^\alpha$?

Theorem 2 Bender [1975]

If $f(x)$ is a power series of a function analytic at the origin, i.e. $|f_n| \leq C^n$, then, for $g \in F_\beta^\alpha$ with $g(0) = 0$:

$$f \circ g \in F_\beta^\alpha$$
$$(\mathcal{A} f \circ g)(x) = f'(g(x))(\mathcal{A} g)(x)$$

- Bender considered much more general power series, but this is a direct corollary of his theorem in 1975.

- What happens if $f \notin \ker \mathcal{A}$?
- $\mathcal{A}$ fulfills a general 'chain rule':

Theorem 3 MB [2016]

If $f, g \in F_\beta^\alpha$ with $g(0) = 0$ and $g'(0) = 1$:

$$f \circ g \in F_\beta^\alpha$$

$$(\mathcal{A}f \circ g)(x) = f'(g(x))(\mathcal{A}g)(x) + \left(\frac{x}{g(x)}\right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}} (\mathcal{A}f)(g(x))$$

$\Rightarrow$ We can solve for asymptotics of implicitly defined power series!

Theorem 3 MB [2016]

If $f, g \in F_\beta^\alpha$ with $g(0) = 0$ and $g'(0) = 1$:

$$f \circ g \in F_\beta^\alpha$$

$$(\mathcal{A} f \circ g)(x) = f'(g(x))(\mathcal{A} g)(x) + \left(\frac{x}{g(x)} \right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}} (\mathcal{A} f)(g(x))$$

- $g'(0) = 1$ not a real restriction. Scaling maps spaces $F_\beta^\alpha \to F_\beta^{\alpha'}$ trivially.
- $e^{\frac{g(x)-x}{\alpha x g(x)}}$ generates 'funny exponentials': Typical prefactors of the form

$$e^{\frac{g_2}{\alpha}}$$

in asymptotic expansions.

Differential equations

$$\partial f(x) = F(f(x), x)$$

with $F(x,y)$ analytic at $(0,0)$.

- Apply $\mathcal{A}$:

$$\mathcal{A}\partial f(x) = \frac{\partial F}{\partial f}(f(x), x)(\mathcal{A}f)(x)$$

- Use $\partial^{\mathcal{A}}$ with $\partial^{\mathcal{A}}\mathcal{A} = \mathcal{A}\partial$:

$$\Rightarrow \partial^{\mathcal{A}}(\mathcal{A}f)(x) = \frac{\partial F}{\partial f}(f(x), x)(\mathcal{A}f)(x)$$

- Linear differential equation for $(\mathcal{A}f)(x)$.

Applications

Action on Dyson-Schwinger-Equations

Let $p, g, f \in F_\beta^\alpha$ and $p \in \ker \mathcal{A}$, then the functional equation,

$$p(g(x)) = x + f(g(x))$$

implies $$(\mathcal{A}g)(x) = g'(x) \left(\frac{x}{g(x)}\right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}} (\mathcal{A}f)(g(x))$$

and $$(\mathcal{A}f)(x) = {g^{-1}}'(x) \left(\frac{x}{g^{-1}(x)}\right)^\beta e^{\frac{g^{-1}(x)-x}{\alpha x g^{-1}(x)}} (\mathcal{A}g)(g^{-1}(x)).$$

where $g(g^{-1}(x)) = x$.

$\Rightarrow$ Solving the DSE ‘perturbativly’ to n terms gives an asymptotic expansion up to order $n-2$!

- $\mathcal{A}$ maps low order expansions to high order expansions.
- Asymptotic expansion independent of p.

Example: Simple permutations

- Let $\pi \in S_n^{\text{simple}} \subset S_n$ such that $\pi([i,j]) \neq [k,l]$ for all $i,j,k,l \in [0,n]$ with $2 \leq |[i,j]| \leq n-1$, then π is a simple permutation, which does not map an interval to another interval.
- With $S(x) = \sum_{n=0}^{\infty} |S_n^{\text{simple}}| x^n$ and $F(x) = \sum_{n=1}^{\infty} n! x^n$:

Albert et al. [2003]

$$\frac{F(x) - F(x)^2}{1 + F(x)} = x + S(F(x))$$

- $F(x) \in F_1^1$ and $(\mathcal{A}F) = 1 \Rightarrow$ even though $S(x)$ is only given implicitly, we have an asymptotic expansion!

- Generating function for asymptotic coefficients of $S(x)$:

$$(\mathcal{A}S)(x) = F^{-1\prime}(x)\left(\frac{x}{F^{-1}(x)}\right)^{\beta} e^{\frac{F^{-1}(x)-x}{\alpha x F^{-1}(x)}}$$

$$s_n = e^{-2} n! \left(1 - \frac{4}{n} + \frac{2}{n(n-1)} - \frac{40}{3n(n-1)(n-2)} + \dots\right)$$

- Generating function for asymptotic coefficients $\Rightarrow$ can analyse asymptotics of asymptotics.

Conclusions

- F_β^α forms a subring of $\mathbb{R}[[x]]$ closed under composition, differentiation* and integration.
- $\mathcal{A}$ is a derivation on F_β^α which can be used to obtain asymptotic expansions of implicitly defined power series.
- Nice closure properties under asymptotic derivative $\mathcal{A}$.
- Generalizations possible to multiple $\alpha_1, \ldots, \alpha_l \in \mathbb{C}$ with $|\alpha_i| = \alpha$.
- Suitable for resummation of perturbation series $\Rightarrow$ applications in QFT and QM!
- There are probably many connections to resurgence!

'Asymptotic calculus'

Action under transformation with $\mathcal{A}$-operator

$$
\begin{aligned}
f(x)g(x) &\rightarrow (\mathcal{A}f)(x)g(x) + f(x)(\mathcal{A}g)(x) \\
\partial f(x) &\rightarrow (\alpha^{-1} - x\beta + x^2\partial)(\mathcal{A}f)(x) \\
f(g(x)) &\rightarrow f'(g(x))(\mathcal{A}g)(x) + \left(\frac{x}{g(x)}\right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}} (\mathcal{A}f)(g(x))
\end{aligned}
$$

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