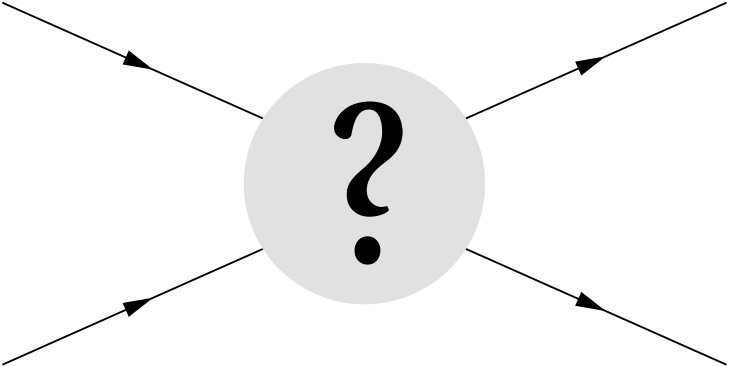
Regularization of hyperlogarithms

Erik Panzer

All Souls College

Paths to, from and in renormalisation
February 9th
Universität Potsdam

Perturbative Quantum Field Theory (**QFT**)



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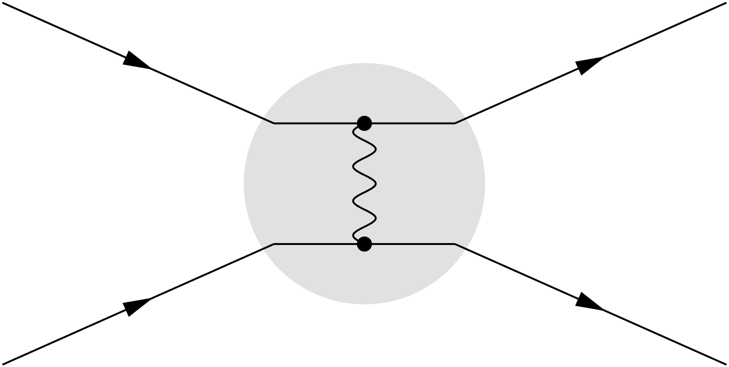

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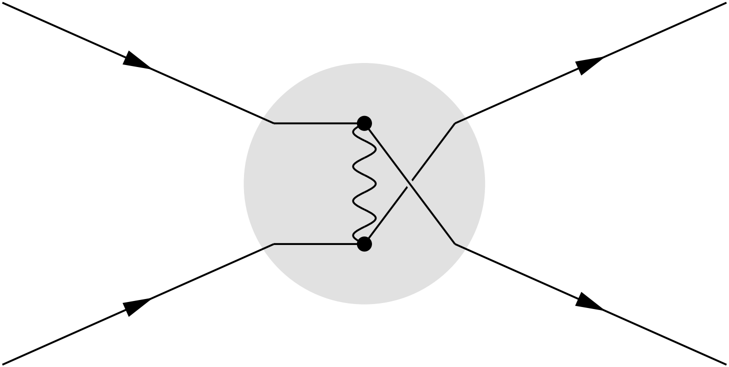

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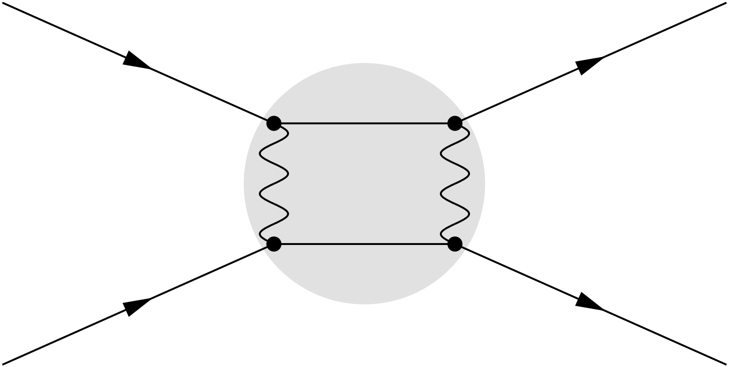

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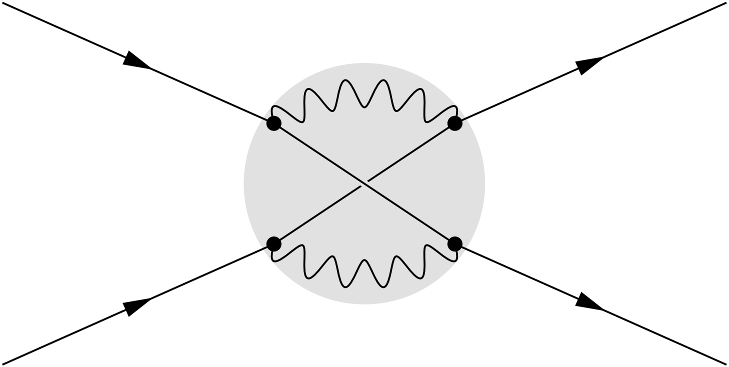

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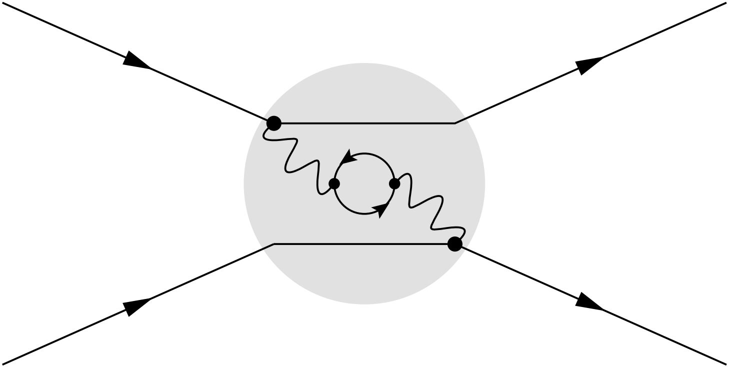

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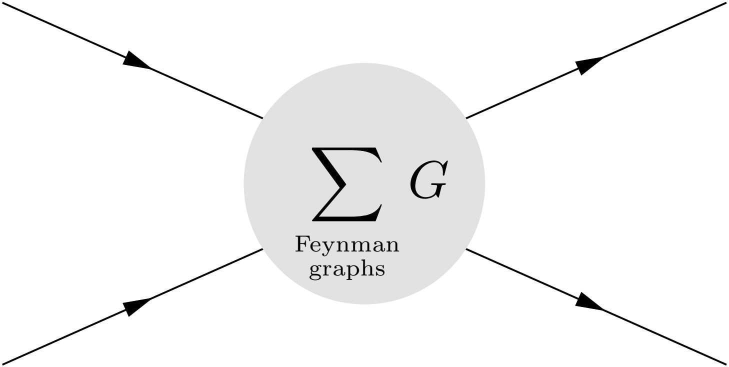

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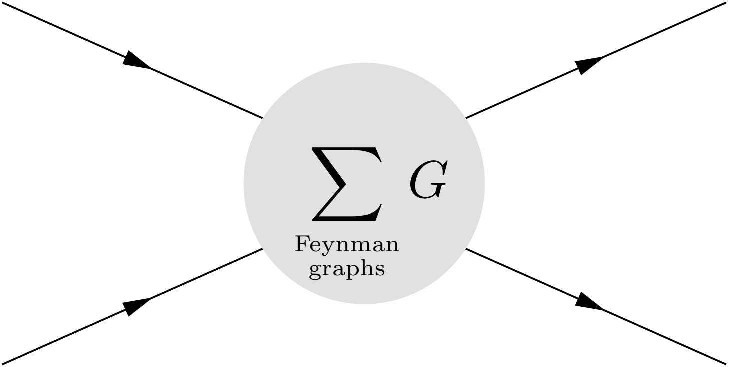

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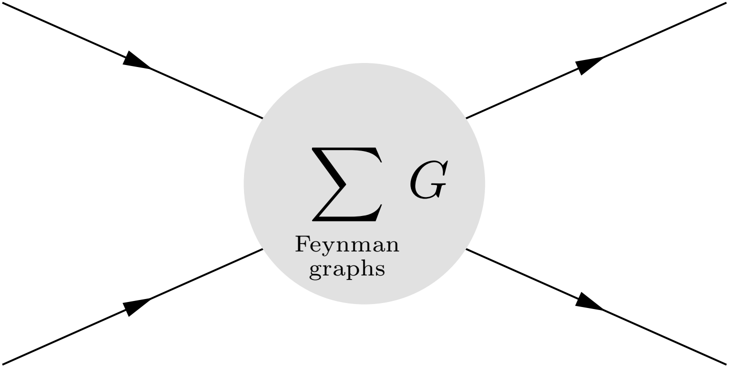
Perturbative Quantum Field Theory (QFT)



- each Feynman graph represents a Feynman integral (FI) $\Phi(G)$
- truncated sum $\sum_G \Phi(G)$ approximates the process
- very accurate measurements demand precise theoretical predictions
  Challenges: **number of graphs** & **complexity of integrals**

Renormalization in QFT

Often, FIs are divergent. Solution:

- (optional) regularization: make FI well-defined via new parameter (cut-off Λ, DimReg $D-4$, ...)
- renormalization: choose a scheme (MS, BPHZ [talk by Chandra], ...)

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Algebraic description:

- Hopf algebras of rooted trees and Feynman diagrams [talks by Bruned, Kreimer],
- Dyson-Schwinger equations [talks by Bellon, Borinsky, Foissy, Klaczynski]
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Warning

Not the topic of this talk!

Some FI are expressible with

- logarithms: $$-\log(1-z) = \sum_{0<k} \frac{z^k}{k}$$
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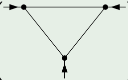
Example

$\Phi\Big($  $\Big)$ $$= \frac{2\,\mathrm{Im}\left[\mathrm{Li}_2(z) + \log(1-z)\log|z|\right]}{\mathrm{Im}\, z}$$

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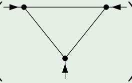

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- their special values, e.g. multiple zeta values (MZV): $$\zeta_{n_1,\ldots,n_d} = \mathrm{Li}_{n_1,\ldots,n_d}(1,\ldots,1)$$

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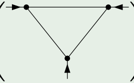
$\Phi\Big($  $\Big)$ $$= \frac{2\,\mathrm{Im}\left[\mathrm{Li}_2(z) + \log(1-z)\log|z|\right]}{\mathrm{Im}\, z}$$

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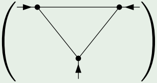
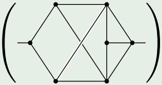
Φ  $$= \frac{2\,\mathrm{Im}\left[\mathrm{Li}_2(z) + \log(1-z)\log|z|\right]}{\mathrm{Im}\, z}$$

Φ  $$= 252\zeta_3\zeta_5 + \tfrac{432}{5}\zeta_{3,5} - \tfrac{25056}{875}\zeta_2^4$$

This talk is about MPL as functions in $z_1, \dots, z_d$,

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Warning

This is very different from the analytic continuation of MZV,

$$\zeta_{n_1,\ldots,n_d} = \mathrm{Li}_{n_1,\ldots,n_d}(1,\ldots,1),$$

in the indices $n_1, \ldots, n_d$, which yields *meromorphic* functions [talks by Singer, Ebrahimi-Fard].

The sum representation for MPL is not useful to understand their analytic continuation. Instead, we want an integral representation:

$$\mathrm{Li}_1(z) = \sum_{0<k} \frac{z^k}{k} = -\log(1-z) = \int_0^z \frac{\mathrm{d}t}{1-t}$$

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Insert these equations into each other (iterated integral):

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iterated integrals (Chen 1973)

Take a manifold X and differential forms $\omega_1, \ldots, \omega_n \in \Omega^1(X)$. Integrating these along a path $\gamma \in C^1([0,1], X)$, we can construct functions (on γ):

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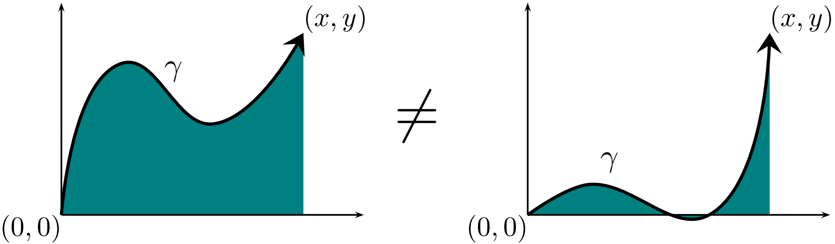

If $\omega = y\mathrm{d}x \in \Omega^1(\mathbb{R}^2)$, then $\int_\gamma \omega$ is the area between γ and the x-axis.

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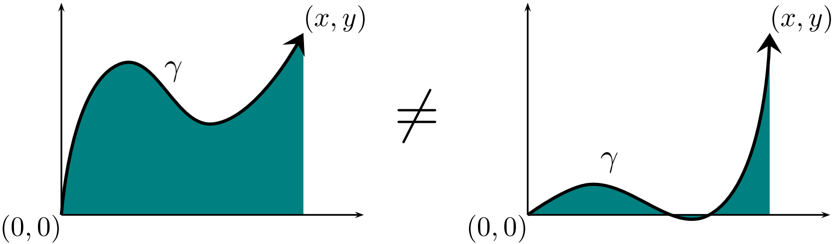

$\Rightarrow$ integrability condition (Chen), simplest case:

$$\int_\gamma \omega \quad \text{homotopy invariant} \Leftrightarrow \mathrm{d}\omega = 0$$

Hyperlogarithms (Poincaré 1884, Lappo-Danilevsky 1927)

Let $X = \mathbb{C} \setminus \Sigma$ for a finite set of points Σ. The regular, non-exact forms

$$\omega_\sigma := \mathrm{d}\log(z-\sigma) = \frac{\mathrm{d}z}{z-\sigma}$$

generate homotopy invariant iterated integrals, called hyperlogarithms.

Examples

$\int_\gamma \omega_0 = \log \frac{\gamma(1)}{\gamma(0)}$, $\int_0^z \omega_1 = \log(1-z)$, $\int_0^z \omega_0\omega_1 = -\int_0^z \frac{\log(1-t)\mathrm{d}t}{t}$

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Notation and some special cases:

- $\int_0^z \omega_{\sigma_n}\cdots\omega_{\sigma_1} = I(0;\sigma_1,\ldots,\sigma_n;z) = G(\sigma_n,\ldots,\sigma_1;z)$ [Goncharov]
- $\Sigma = \{-1,0,1\}$ harmonic polylogarithms (HPL) [Remiddi & Vermaseren]
- $\Sigma = \{0,1,1-y,-y\}$ 2-dimensional HPL [Gehrmann & Remiddi]

Consider the monoid of all words over Σ,

$$\Sigma^* = 1 \,\dot{\cup}\, \{\omega_\sigma \colon\ \sigma \in \Sigma\} \,\dot{\cup}\, \{\omega_{\sigma_1}\omega_{\sigma_2} \colon\ \sigma_1, \sigma_2 \in \Sigma\} \,\dot{\cup} \cdots$$

where 1 denotes the *empty word*. Let the tensor algebra

$$T(\Sigma) := \mathbb{Q}\langle \omega_\sigma \colon \sigma \in \Sigma \rangle = \mathrm{lin}_{\mathbb{Q}}\, \Sigma^*$$

denote their linear combinations. We extend $\int_\gamma$ to a linear functional on $T(\Sigma)$, setting $\int_\gamma 1 := 1$.

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$$T(\Sigma) := \mathbb{Q}\langle \omega_\sigma\colon \sigma \in \Sigma\rangle = \operatorname{lin}_{\mathbb{Q}} \Sigma^*$$

denote their linear combinations. We extend $\int_\gamma$ to a linear functional on $T(\Sigma)$, setting $\int_\gamma 1 := 1$. By construction, for all $w \in T(\Sigma)$ it obeys

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How does $\int_\gamma$ behave with respect to the Hopf algebra structure of $T(\Sigma)$?

- shuffle product
- deconcatenation coproduct

Shuffle product

The shuffle product of two words

$$w_{n+m} \cdots w_{n+1} \sqcup\!\sqcup\, w_n \cdots w_1 = \sum_{\sigma} w_{\sigma(n+m)} \cdots w_{\sigma(1)}$$

is the sum of all their shuffles σ, i.e. permutations which preserve the relative order of letters in both factors:

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(n) \quad \text{and} \quad \sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m).$$

For arbitrary words u and v, we find that

$$\left(\int_{\gamma} u\right) \cdot \left(\int_{\gamma} v\right) = \int_{\gamma} (u \sqcup\!\sqcup\, v).$$

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Example

$$\int_{\gamma} \omega_3 \cdot \int_{\gamma} \omega_2 \omega_1 =$$

$$\{t_3\} \times \{t_1 \leq t_2\} = \{t_1 \leq t_2 \leq t_3\} \cup \{t_1 \leq t_3 \leq t_2\} \cup \{t_3 \leq t_1 \leq t_2\}$$

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$$\int_\gamma \omega_3 \cdot \int_\gamma \omega_2\omega_1 = \int_\gamma \omega_3\omega_2\omega_1 + \int_\gamma \omega_2\omega_3\omega_1 + \int_\gamma \omega_2\omega_1\omega_3$$

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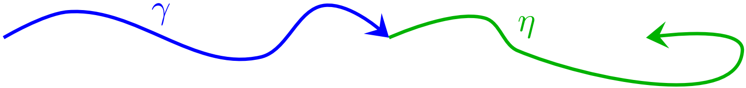
Path concatenation

Let $\gamma \star \eta$ denote the concatenation of γ and η at $\gamma(1) = \eta(0) = (\gamma \star \eta)(\frac{1}{2})$:



To decompose

$$\int_{\gamma \star \eta} \omega_2 \omega_1 = \int_{0 \le t_1 \le t_2 \le 1} (\gamma \star \eta)^*(w_2)(t_2)(\gamma \star \eta)^*(w_1)(t_1),$$

split the interval

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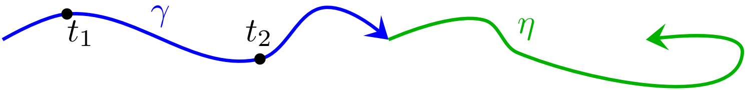

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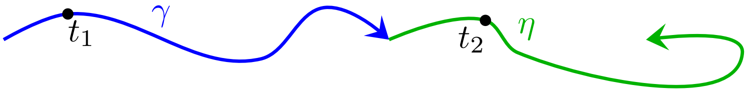

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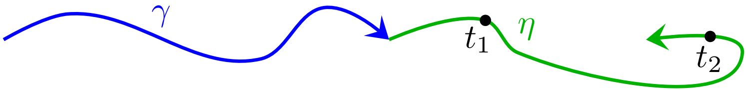

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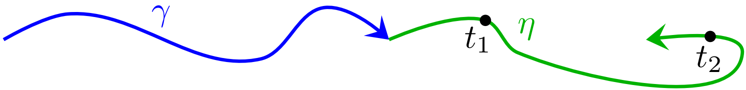

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$$\int_{\gamma\star\eta} \omega_2\omega_1 = \int_\gamma \omega_2\omega_1 + \int_\eta \omega_2 \int_\gamma \omega_1 + \int_\eta \omega_2\omega_1,$$

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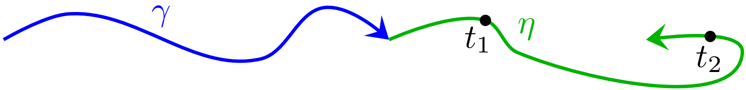

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More generally, the path concatenation formula reads

$$\int_{\gamma\star\eta} \omega_n \cdots \omega_1 = \sum_{k=0}^{n} \int_\eta \omega_n \cdots \omega_{k+1} \int_\gamma \omega_k \cdots \omega_1.$$

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Recall

Inverses of characters can be computed with the antipode:

$$\psi^{-1} = \psi \circ S \qquad (\text{such that } \psi \star \psi^{-1} = \varepsilon)$$

where $S(\omega_1 \cdots \omega_n) = (-\omega_n) \cdots (-\omega_1)$.

If $\gamma \simeq \gamma(0)$ is homotopic to a constant path, then $\int_\gamma = \varepsilon$. Hence

$$\varepsilon = \int_\gamma \star \int_{\gamma^{-1}} \quad \text{so} \quad \left(\int_\gamma\right)^{-1} = \int_{\gamma^{-1}} = \int_\gamma S$$

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Changing base points

$$\int_a^z = \int_b^z \star \int_a^b, \quad \text{and} \quad \int_a^b = \int_b^z S \star \int_a^z$$

is mediated by (right-) convolution with a constant character $\int_a^b$.

Singularities

Singularities when $\{\gamma(0), \gamma(1)\} \ni z \to \tau \in \{\infty\} \cup \Sigma$ are logarithmic: There exist (uniquely determined) functions $f_{k,w}(z)$, analytic at $z = \tau$, such that

$$\int_\gamma w = \sum_{k \geq 0} \log^k(z-\tau) f_{k,w}(z) \quad (\text{finite sum})$$

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Definition (logarithmic regularization)

The regularized limit is $\operatorname{Reg}_{z \to \tau} \int_\gamma w := f_{0,w}(\tau)$.

Example

$$\int_b^z \omega_0 = \log \frac{z}{b} \qquad \int_0^z \omega_0 := \operatorname*{Reg}_{b \to 0} \int_b^z \omega_0 = \log(z)$$

Shuffle regularization

If the last letter of w is not ω_0, then $\int_0^z w$ converges:

$$\operatorname*{Reg}_{b\to 0}\int_b^z w = \lim_{b\to 0}\int_b^z w = \int_0^z w$$

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For every $w \in T(\Sigma)$ there are unique $w_k \in T(\Sigma)$ not ending in ω_0 so that

$$w = \sum_{k\geq 0} w_k \sqcup\!\sqcup\, \omega_0^k$$

Example

$$\omega_0\omega_1\omega_0 = \underbrace{\omega_0\omega_1}_{w_1} \sqcup\!\sqcup\, \omega_0 \underbrace{-2\omega_0\omega_0\omega_1}_{w_0}$$

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$$\operatorname{reg}_0\left(\sum_{k\geq 0} w_k ⧢ \omega_0^k\right) := w_0$$

defines a character and projection onto words not ending in ω_0.

One can show that, for $w = u\omega_\sigma\omega_0^n$ with $\sigma \neq 0$,

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Lemma

Let $P_\sigma\colon T(\Sigma) \twoheadrightarrow T(\{\sigma\})$ denote the projection onto powers of ω_σ, then

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Similarly, we can write $w = \sum_{k\geq 0} w_k ⧢ \omega_1^k$ such that w_k does not begin with 1 and set $\operatorname{reg}^1(w) := w_0$.

Definition (Shuffle-regularized MZV)

$$\zeta_{⧢}(n_1,\ldots,n_d) := (-1)^d \int_0^1 \operatorname{reg}^1\left(\omega_0^{n_d-1}\omega_1\cdots\omega_0^{n_1-1}\omega_1\right)$$

Monodromies

Remember

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Example

How does $\mathrm{Li}_2(z) = -\int_0^z \omega_0\omega_1$ behave near 1?

$$\begin{aligned}\mathrm{Li}_2(z) &= \mathrm{Li}_2(1) - \int_1^z \omega_0 \int_0^z \omega_1 + \int_1^z \omega_1\omega_0 \\ &= \zeta_2 - \log(z)\log(1-z) - \mathrm{Li}_2(1-z)\end{aligned}$$

Monodromies

Remember

Even though γ is suppressed in the notation and $\int_0^z w$, these functions still depend on the homotopy class of γ.

$$\int_0^z = \int_1^z \mathrm{reg}_1 \star \int_0^z P_1 \star \int_0^1 \mathrm{reg}^1$$

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Analytic continuation by encircling 1 contributes $\pm 2\pi i \log(z)$.

Note: This only adds lower weight functions.