An index theorem for compact Lorentzian manifolds with boundary

Christian Bär
(joint w. Alexander Strohmaier)

Institut für Mathematik
Universität Potsdam

Paths to, from and in renormalization
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Atiyah(-Patodi)-Singer Index Theorem

Lorentzian manifolds

Atiyah-Singer Index Theorem

- M Riemannian manifold, compact, without boundary
- spin structure $\rightsquigarrow$ spinor bundle $SM \to M$
- $n = \dim(M)$ even $\rightsquigarrow$ splitting $SM = S_R M \oplus S_L M$
- Hermitian vector bundle $E \to M$ with connection $\rightsquigarrow$ twisted Dirac operator $D : C^\infty(M, V_R) \to C^\infty(M, V_L)$ where $V_{R/L} = S_{R/L} M \otimes E$

Theorem (M. Atiyah, I. Singer, 1968)

The operator D is Fredholm and

$$\operatorname{ind}(D) = \int_M \widehat{\mathrm{A}}(M) \wedge \operatorname{ch}(E)$$

Boundary Conditions

Now let M have nonempty boundary.
Need boundary conditions:
Choose "Fermi coordinate function" $t : M \to \mathbb{R}$ and write

$$D = \gamma \left(\frac{\partial}{\partial t} + A_t \right)$$

A_0 is a selfadjoint Dirac-type operator on ∂M.
$P_+ = \chi_{[0,\infty)}(A_0) =$ spectral projector

APS-boundary conditions:

$$P_+(f|_{\partial M}) = 0$$

Atiyah-Patodi-Singer index theorem

Theorem (M. Atiyah, V. Patodi, I. Singer, 1975)

Under APS-boundary conditions D is Fredholm and



$$\mathrm{ind}(D_{\mathrm{APS}}) = \int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) + \int_{\partial M} T(\widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)) - \frac{h(A_0) + \eta(A_0)}{2}$$

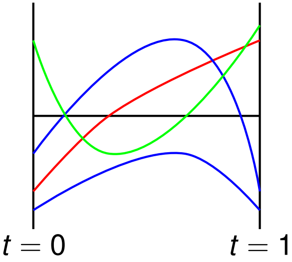
Here

- $h(A) = \dim \ker(A)$
- $\eta(A) = \eta_A(0)$ where $\eta_A(s) = \sum_{\substack{\lambda \in \mathrm{spec}(A) \\ \lambda \neq 0}} \mathrm{sign}(\lambda) \cdot |\lambda|^{-s}$

Spectral flow

Special case $M = \Sigma \times [0,1]$ and $g = dt^2 + g_t$
Then $D = \gamma\left(\frac{\partial}{\partial t} + A_t\right)$ holds globally and

$$\mathrm{sf}(A_{t\in[0,1]}) = \mathrm{ind}(D_{\mathrm{APS}}) + h(A_1)$$



Warning

APS-boundary conditions cannot be replaced by anti-Atiyah-Patodi-Singer boundary conditions,

$$P_-(f|_{\partial M}) = \chi_{(-\infty,0)}(A_0)(f|_{\partial M}) = 0$$

Example

- M = unit disk $\subset \mathbb{C}$
- $D = \overline{\partial} = \frac{\partial}{\partial \overline{z}}$
- Fourier expansion: $u|_{\partial M} = \sum_{n\in\mathbb{Z}} \alpha_n e^{in\theta}$
- $A_0 = i\frac{d}{d\theta}$
- Taylor expansion: $u = \sum_{n=0}^{\infty} \alpha_n z^n$

APS-boundary conditions:
$\alpha_n = 0$ for $n \geq 0 \Rightarrow \ker(D) = \{0\}$
aAPS-boundary conditions:
$\alpha_n = 0$ for $n < 0 \Rightarrow \ker(D) =$ infinite dimensional

Lorentzian manifolds

Replace “spaces” by “spacetimes”,
i.e. Riemannian manifolds by Lorentzian manifolds.

Dirac operator no longer elliptic, but hyperbolic.
In particular, no elliptic regularity theory

Compactness?

Problem 1: Compact Lorentzian manifolds (without boundary) violate causality conditions
$\Rightarrow$ useless as models in General Relativity

Problem 2: hyperbolic PDE theory does not work on such spacetimes
$\Rightarrow$ no Lorentzian analog to Atiyah-Singer index theorem

Globally hyperbolic spacetimes

A subset $\Sigma \subset M$ is called Cauchy hypersurface if each inextensible timelike curve in M meets Σ precisely once.

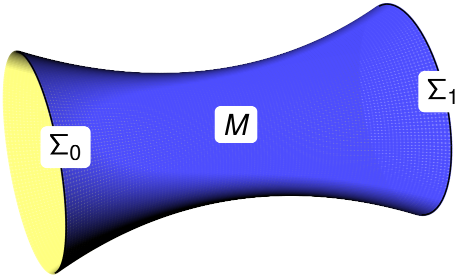
If M has a Cauchy hypersurface then M is called globally hyperbolic.

Examples:

- Minkowski spacetime (Special Relativity)
- Schwarzschild Model (Black Hole)
- Friedmann cosmos (Big Bang, cosmic expansion)
- deSitter spacetime
- $\ldots$

The Lorentzian index theorem

Let M be a compact globally hyperbolic Lorentzian manifold
with boundary $\partial M = \Sigma_0 \sqcup \Sigma_1$
Σ_j smooth spacelike Cauchy hypersurfaces
D twisted Dirac operator



The Lorentzian index theorem

Theorem (C. B., A. Strohmaier, 2015)

Under APS-boundary conditions D is a Fredholm operator. The kernel consists of smooth spinor fields and

$$\operatorname{ind}(D_{\mathrm{APS}}) = \int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) + \int_{\partial M} T(\widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)) - \frac{h(A_0) + h(A_1) + \eta(A_0) - \eta(A_1)}{2}$$

Moreover,

$$\operatorname{ind}(D_{\mathrm{APS}}) = \dim\ker[D : C^\infty_{\mathrm{APS}}(M; V_R) \to C^\infty(M; V_L)] - \dim\ker[D : C^\infty_{\mathrm{aAPS}}(M; V_R) \to C^\infty(M; V_L)]$$

aAPS conditions are as good as APS-boundary conditions

Proof of the regularity statement

- If Φ is a distributional spinor solving $D\Phi = 0$ then $\mathrm{WF}(\Phi) \subset \{\text{lightlike covectors}\}$
- Φ restricts to distributions along $\Sigma_{0/1}$
- APS conditions along $\Sigma_0 \Rightarrow$ $\mathrm{WF}(\Phi) \subset \{\text{future-directed lightlike covectors}\}$ along Σ_0
- propagation of singularities $\Rightarrow$ $\mathrm{WF}(\Phi) \subset \{\text{future-directed lightlike covectors}\}$ on all of M
- similarly, APS along $\Sigma_1 \Rightarrow$ $\mathrm{WF}(\Phi) \subset \{\text{past-directed lightlike covectors}\}$
- $\Rightarrow \mathrm{WF}(\Phi) = \emptyset$, i.e. Φ is smooth

Application to physics: the chiral anomaly in QFT

No natural physical interpretation of APS boundary conditions in the Riemannian case.
But the Lorentzian version allows to compute the chiral anomaly in QFT.

References:

C. Bär and A. Strohmaier: *An index theorem for Lorentzian manifolds with compact spacelike Cauchy boundary*
`arXiv:1506.00959`

C. Bär and A. Strohmaier: *A rigorous geometric derivation of the chiral anomaly in curved backgrounds*
`arXiv:1508.05345`

Thanks for your attention!