Renormalization and Euler-Maclaurin Formula on Cones

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Outline

- Conical zeta values and multiple zeta values;
- Double shuffle relations and double subdivision relations;
- Renormalization of conical zeta values;
- Euler-Maclaurin formula.
Cones

- A (closed polyhedral) cone in $\mathbb{R}^k_{\geq 0}$ is defined to be the convex set
  \[
  \langle v_1, \cdots, v_n \rangle := \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n, \quad v_i \in \mathbb{R}^k_{\geq 0}, \quad 1 \leq i \leq n.
  \]

- The interior of a cone $\langle v_1, \cdots, v_n \rangle$ is an open (polyhedral) cone
  \[
  \langle v_1, \cdots, v_n \rangle^o := \mathbb{R}_{> 0} v_1 + \cdots + \mathbb{R}_{> 0} v_n.
  \]

- The set $\{v_1, \cdots, v_n\}$ is called the generating set or the spanning set of the cone. The dimension of a cone is the dimension of linear subspace generated by it.

- Let $\mathcal{C}_k$ (resp. $\mathcal{O}\mathcal{C}_k$) denote the set of closed (resp. open cones) in $\mathbb{R}^k$, $k \geq 1$. For $k = 0$ we set $\mathcal{C}_0 = \{0\}$ (resp. $\mathcal{O}\mathcal{C}_0 = \{0\}$) by convention. Through the natural inclusions $\mathcal{C}_k \rightarrow \mathcal{C}_{k+1}$ (resp. $\mathcal{O}\mathcal{C}_k \rightarrow \mathcal{O}\mathcal{C}_{k+1}$) from the natural inclusion $\mathbb{R}^k \rightarrow \mathbb{R}^{k+1}$, we define $\mathcal{C} = \lim_{\rightarrow} \mathcal{C}_k$ (resp. $\mathcal{O}\mathcal{C} = \lim_{\rightarrow} \mathcal{O}\mathcal{C}_k$).
A simplicial cone is defined to be a cone spanned by linearly independent vectors.

A rational cone is a cone spanned by vectors in \( \mathbb{Z}^k \subseteq \mathbb{R}^k \).

A smooth cone is a rational cone with a spanning set that is a part of a basis of \( \mathbb{Z}^k \subseteq \mathbb{R}^k \). In this case, the spanning set is unique and is called the primary set of the cone.

A cone is called strongly convex or pointed if it does not contain any linear subspace.

A subdivision of a closed cone \( C \in \mathcal{C}_k \) is a set \( \{ C_1, \cdots, C_r \} \subseteq \mathcal{C}_k \) such that \( C = \bigcup_{i=1}^r C_i \), \( C_1, \cdots, C_r \) have the same dimension \( C \) and intersect along their faces. The faces of the relative interior give an open subdivision of \( C^o \):

\[
\langle e_1, e_2 \rangle = \langle e_1, e_1 + e_2 \rangle \cup \langle e_1 + e_2, e_2 \rangle
\]

\[
\Rightarrow \langle e_1, e_2 \rangle^o = \langle e_1, e_1 + e_2 \rangle^o \cup \langle e_1 + e_2, e_2 \rangle^o \cup \langle e_1 + e_2 \rangle^o.
\]

For \( \vec{x} = (x_1, \cdots, x_k) \) and \( \vec{y} = (y_1, \cdots, y_k) \) in \( \mathbb{R}^k \), let \((\vec{x}, \vec{y})\) denote the inner product \( x_1 y_1 + \cdots + x_k y_k \). Through this inner product, \( \mathbb{R}^k \) is identified with its own dual space \((\mathbb{R}^k)^*\).
Conical zeta values

- Let $C$ be a smooth cone. The conical zeta function of $C$ is
  \[ \zeta(C; \vec{s}) := \sum_{(n_1, \ldots, n_k) \in C^o \cap \mathbb{Z}^k} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}, \quad \vec{s} \in \mathbb{C}^k, \]
  if the sum converges. When $s_i, 1 \leq i \leq k$, are integers, $\zeta(\vec{s})$ is called a conical zeta value (CZV). Convention: $0^s = 1$ for any $s$. Hence $\zeta(\vec{s})$ does not depend on the choice of $k$.
- If $s_i \geq 2, 1 \leq i \leq k$, then $\zeta(C; \vec{s})$ converges.
- If $\{C_i\}_i$ is an open cone subdivision of $C$, then
  \[ \zeta(C; \vec{s}) = \sum_i \zeta(C_i; \vec{s}). \]

- The cone subdivision
  \[ \langle e_1, e_2 \rangle^o = \langle e_1, e_1 + e_2 \rangle^o \sqcup \langle e_1 + e_2, e_2 \rangle^o \sqcup \langle e_1 + e_2 \rangle^o \]
  gives
  \[ \zeta(\langle e_1, e_2 \rangle^o; (s_1, s_2)) = \zeta(\langle e_1, e_1 + e_2 \rangle^o; (s_1, s_2)) \]
  \[ + \zeta(\langle e_1 + e_2, e_2 \rangle^o; (s_1, s_2)) + \zeta(\langle e_1 + e_2 \rangle^o; (s_1, s_2)). \]
Chen cones and multiple zeta values

A Chen cone of dimension $k$ is a cone

$$C_{k,\sigma} := \langle e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)}, \ldots, e_{\sigma(1)} + \cdots + e_{\sigma(k)} \rangle,$$

where $\sigma \in S_k$. Let $C_k$ denote the standard Chen cone spanned by $\{e_1, \cdots, e_k\}$.

Then $\zeta(C_{k,\sigma}; s_1, \cdots, s_k) = \zeta(s_{\sigma(1)}, \cdots, s_{\sigma(k)})$,

$$\zeta(C_{k,\text{id}}; s_1, \cdots, s_k) = \zeta(s_1, \cdots, s_k).$$

The stuffle product of two MZVs $\zeta(r_1, \cdots, r_k)$ and $\zeta(s_1, \cdots, s_\ell)$ is recovered by the subdivision of the cone $C_k \times C_\ell$ (direct product) into Chen cones.

For example, the open cone subdivision relation

$$\zeta(\langle e_1, e_2 \rangle^o; (s_1, s_2)) = \zeta(\langle e_1, e_1 + e_2 \rangle^o; (s_1, s_2))$$

$$+ \zeta(\langle e_1 + e_2, e_2 \rangle^o; (s_1, s_2) + \zeta(\langle e_1 + e_2 \rangle^o; (s_1, s_2))$$

gives the stuffle relation

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).$$
Multiple zeta values

- The multiple zeta value algebra is
  \[
  \text{MZV} := \mathbb{Q}\{\zeta(s_1, \ldots, s_k) \mid s_i \geq 1, s_1 \geq 1\}.
  \]

- The quasi-shuffle algebra \(\mathcal{H}^*\) has the underlying vector space
  \[
  \mathbb{Q}\langle z_s \mid s \geq 1 \rangle
  \]
  with the quasi-shuffle product. It contains the subalgebra
  \[
  \mathcal{H}_0^* := \mathbb{Q}.1 \oplus \left( \bigoplus_{s_1 \geq 2} \mathbb{Q}z_{s_1} \cdots z_{s_k} \right) \subseteq \mathcal{H}^*.
  \]

The stuffle relation of MZVs is encoded in the algebra homomorphism
\[
\zeta^* : \mathcal{H}_0^* \longrightarrow \text{MZV}, \quad z_{s_1} \cdots z_{s_k} \mapsto \zeta(s_1, \ldots, s_k).
\]
The shuffle algebra \( \mathcal{H}^{III} \) has the underlying vector space \( \mathbb{Q}\langle x_0, x_1 \rangle \) equipped with the shuffle product of words. It contains the subalgebra
\[
\mathcal{H}^{III}_0 := \mathbb{Q}.1 \bigoplus x_0 \mathcal{H}^{III} x_1.
\]
The shuffle relation of the MZVs is encoded in the algebra homomorphism
\[
\zeta^{III} : \mathcal{H}^{III}_0 \to \text{MZV}, \quad x_0^{s_1-1} x_1 \cdots x_0^{s_k-1} x_1 \mapsto \zeta(s_1, \cdots, s_k).
\]
There is a natural bijection of abelian groups (but not algebras)
\[
\eta : \mathcal{H}^{III}_0 \to \mathcal{H}^*, \quad 1 \leftrightarrow 1, \quad x_0^{s_1-1} x_1 \cdots x_0^{s_k-1} x_1 \leftrightarrow z_{s_1} \cdots z_{s_k}.
\]
Then the fact that MZVs can be multiplied in two ways is reflected by
\[
\mathcal{H}^{III}_0 \xrightarrow{\eta} \mathcal{H}^* \xleftarrow{\zeta^*} \mathcal{H}^{III}_0 \xrightarrow{\zeta^{III}} \text{MZV}.
\]
Double shuffle relation
\[
\zeta^*(w_1 \ast w_2 - \eta(\eta^{-1}(w_1)^{III} \eta^{-1}(w_2))), \quad w_1, w_2 \in \mathcal{H}^*_0.
\]
Let $\langle v_1, \cdots, v_k \rangle$ be a smooth close cone with its (unique) primitive generating set.

For $s_1, \cdots, s_k \geq 1$, called the formal expression $[v_1]^{s_1} \cdots [v_k]^{s_k}$ a decorated smooth cone.

Define the linearly constrained zeta value (LZV)

$$
\zeta^c([v_1]^{s_1} \cdots [v_k]^{s_k}) := \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{1}{(a_{11} m_1 + \cdots + a_{1r} m_r)^{s_1} \cdots (a_{k1} m_1 + \cdots + a_{kr} m_r)^{s_k}}
$$

if the sum is convergent, where $v_i = \sum_{j=1}^{r} a_{ij} e_j$, $1 \leq i \leq k$. When $[v_1] \cdots [v_k]$ is a Chen cone $[e_1] \cdots [e_1 + \cdots + e_k]$, then we have

$$
\zeta^c([v_1]^{s_1} \cdots [v_k]^{s_k}) = \zeta(s_1, \cdots, s_k).
$$
Subdivision of decorated closed cones

Let \( \{ \langle v_1, \ldots, v_k \rangle \} \) be a smooth subdivision of the smooth cone \( \langle v_1, \ldots, v_k \rangle \). Call \( \sum_i [v_{i1}] \cdots [v_{ik}] \) an algebraic subdivision of \( [v_1] \cdots [v_k] \).

Let \( [v_1]^{s_1} \cdots [v_k]^{s_k} \) be a decorated smooth closed cone.

Define \( \delta_{e_i}([v_1]^{s_1} \cdots [v_k]^{s_k}) = \sum_j s_j(e_i, v_j)[v_1]^{s_1} \cdots [v_j]^{s_j+1} \cdots [v_k]^{s_k} \). For \( u = \sum_i c_i e_i \), define \( \delta_u = \sum_i c_i \delta_{e_i} \). Then \( [v_1]^{s_1} \cdots [v_k]^{s_k} = \frac{1}{(s_1-1)! \cdots (s_k-1)!} \delta_{v_1^*}^{s_1-1} \cdots \delta_{v_k^*}^{s_k-1}([v_{i1}] \cdots [v_{ik}]) \).

Call

\[
\sum_i \frac{1}{(s_1-1)! \cdots (s_k-1)!} \delta_{v_1^*}^{s_1-1} \cdots \delta_{v_k^*}^{s_k-1}([v_{i1}] \cdots [v_{ik}])
\]

an algebraic subdivision of \( [v_1]^{s_1} \cdots [v_k]^{s_k} \). Here \( v_1^*, \ldots, v_k^* \) is a dual basis of \( v_1, \ldots, v_k \).

Let \( D = \sum_i a_i D_i \) be an algebraic subdivision of a decorated smooth cone \( D \). Then

\[
\zeta^c(D) = \sum_i a_i \zeta^c(D_i).
\]

This generalizes the shuffle relation of MZVs.
Let $GL_r(\mathbb{Z})$ denote the set of $r \times r$ unimodular matrices. Let $M \in GL_r(\mathbb{Z})$ and $\vec{s} := (s_1, \ldots, s_r) \in \mathbb{Z}_r^{\geq 0}$. Let $v_1, \ldots, v_r$ and $u_1, \ldots, u_r$ be the row and column vectors of $M$. The (decorated) cone pair associated with $M$ and $\vec{s}$ is the pair $(C, D)$ consisting of the decorated open cone $C := C_{M, \vec{s}} = (\langle u_1, \cdots, u_r \rangle^o, \vec{s})$ and the decorated closed cone $D := D_{M, \vec{s}} = [v_1]^{s_1} \cdots [v_r]^{s_r}$. We call the pair convergent if the corresponding $\zeta$-values $\zeta^0(C)$ and $\zeta^c(D)$ converge.

Let $\mathcal{DTP}$ denote the set of cone pairs $(C_{M, \vec{s}}, D_{M, \vec{s}})$ where $M \in O(\mathbb{Z})$ and $\vec{s} \in \mathbb{Z}_r^{\geq 0}$. Let

$$p^O : \mathbb{QDTP} \to \mathbb{QDC}$$

and

$$p^C : \mathbb{QDTP} \to \mathbb{QDMC}$$

denote the natural projections.

For any cone pair $(C, D) \in \mathcal{DTP}$, we have

$$\zeta^O(C) = \zeta^c(D),$$

if either side makes sense.
Double subdivision relation

Let \((C, D)\) be a convergent cone pair. Let \(\{C_i\}_i\) be an open subdivision of the decorated open cone \(C\) and let \(\sum_j c_j D_j\) be a subdivision of the decorated closed cone \(D\). Also let \(D_j^T \in \mathcal{DC}\) be the transpose cone of \(D_j\), that is, \((D_j^T, D_j)\) is a cone pair. Then

\[
\sum_i C_i - \sum_j c_j D_j^T
\]

(1)

lies in the kernel of \(\zeta^o\). It is called a double subdivision relation.

For any not necessarily convergent cone pair \((C, D)\), let \(\{C_i\}\) be a subdivision of \(C\) and \(\sum_j a_j D_j\) a subdivision of \(D\). If \(\sum_i C_i - \sum_j a_j D_j^T\) is in \(\mathcal{QDc}\), then it is called an extended double subdivision relation.

Hunch. The kernel of \(\zeta^o\) is the subspace \(I_{EDS}\) of \(\mathcal{QDc}\) generated by the extended double subdivision relations.
Double subdivision relation
Algebraic Birkhoff Decomposition. Let \( \mathcal{H} \) be a connected filtered Hopf algebra, \( R = P(R) \oplus (\text{id} - P)(R) \) a commutative Rota-Baxter algebra with an idempotent Rota-Baxter operator \( P \). Any algebra homomorphism \( \phi : \mathcal{H} \rightarrow R \) has a unique decomposition into algebra homomorphisms

\[
\phi = \phi_{-1} \ast \phi_+,
\]

\[
\begin{align*}
\phi_- : \mathcal{H} &\rightarrow \mathbb{C} + P(R) \text{ (counter term)} \\
\phi_+ : \mathcal{H} &\rightarrow \mathbb{C} + (\text{id} - P)(R) \text{ (renormalization)}
\end{align*}
\]

\[\phi_+ : \mathcal{H} \rightarrow \mathbb{C} + (\text{id} - P)(R) \text{ (renormalization)}\]
In QFT renormalization (Dim-Reg scheme), we take the triple $(\mathcal{H}_{FG}, R_{FG}, \phi_{FG})$ with

- Hopf algebra $\mathcal{H}_{FG}$ of Feynman graphs;
- $R_{FG} = \mathbb{C}[\varepsilon^{-1}, \varepsilon]$ of Laurent series, with the pole part projection $P$;
- $\phi_{FG} : \mathcal{H}_{FG} \rightarrow R_{FG}$ from dimensional regularized Feynman rule.

Then Algebraic Birkhoff Decomposition gives

$$\phi_{FG} = \phi_{FG,-} \ast \phi_{FG,+}$$

\[ \begin{array}{c}
\mathcal{H}_{FG} \\
\phi_{FG} \\
\mathbb{C}[\varepsilon^{-1}, \varepsilon]] \\
\end{array} \xrightarrow{\text{Feynman rules}} \begin{array}{c}
\text{Feynman integrals}(= \infty!)
\\
\phi_{FG,+}
\\
\mathbb{C}[[\varepsilon]]
\\
\varepsilon \rightarrow 0
\end{array} \]
Let $\mathbf{C} = \bigoplus_{n \geq 0} \mathbf{C}^{(n)}$ be a (co)differential connected coalgebra (so $\mathbf{C}^{(0)} = kJ$) with counit $\varepsilon : \mathbf{C} \to k$ and coderivations $\delta_\sigma, \sigma \in \Sigma$. Let $A$ be a differential algebra with derivations $\partial_\sigma, \sigma \in \Sigma$. Let $A = A_1 \oplus A_2$ be a linear decomposition such that $1_A \in A_1$ and

$$
\partial_\sigma(A_i) \subseteq A_i, \quad i = 1, 2, \quad \sigma \in \Sigma.
$$

Let $P$ be the projection of $A$ to $A_1$ along $A_2$. Denote

$$
\mathcal{G}(\mathbf{C}, A) := \{ \phi : \mathbf{C} \to A \mid \phi(J) = 1_A, \partial_\sigma \phi = \phi \delta_\sigma, \sigma \in \Sigma \}.
$$

Then any $\phi \in \mathcal{G}(\mathbf{C}, A)$ has a unique decomposition

$$
\varphi = \varphi_1^{*(-1)} \ast \varphi_2,
$$

where $\varphi_i \in \mathcal{G}(\mathbf{C}, A), i = 1, 2$, satisfy $(\ker \varepsilon) \subseteq A_i$ (hence $\varphi_i : \mathbf{C} \to k1_A + A_i$). If moreover $A_1$ is a subalgebra of $A$ then $\phi_1^{*(-1)}$ lies in $\mathcal{G}(\mathbf{C}, A_1)$.
Transverse cones

- Identify $V_k := \mathbb{R}^k$ with its dual through a fixed inner product $(\cdot, \cdot)$.
- For a cone $C$, let $\text{lin}(C)$ denote the subspace spanned by $C$.
- For any closed cone $C$ and its face $F$, define the transverse cone (Berline and Vergne) $t(C, F)$ along $F$ to be the projection of $C$ to $F^\perp$, where $F^\perp = \text{lin}_C^\perp(F)$ is the orthogonal completion of $\text{lin}(F)$ in $\text{lin}(C)$.
- For example, the transverse cone of $\langle e_1, e_1 + e_2 \rangle$ along $\langle e_1 + e_2 \rangle$ is $\langle e_1 - e_2 \rangle$. 
Coproduct of cones

We equip the linear space \( \mathcal{QC} \) of close cones with a coproduct

\[
\Delta : \mathcal{QC} \rightarrow \mathcal{QC} \otimes \mathcal{QC}, \quad \Delta C := \sum_{F \leq C} t(C, F) \otimes F
\]

and a counit

\[
\varepsilon : \mathcal{QC} \rightarrow \mathbb{Q}, \quad \varepsilon(C) = \begin{cases} 
1, & C = \{0\}, \\
0, & C \neq \{0\}.
\end{cases}
\]

With \( \mathcal{CC}^{(n)} := \{ C \in \mathcal{CC} \mid \dim C = n \}, \ n \geq 0 \), we have a connected coalgebra

\[
\mathcal{CC} = \oplus_{n \geq 0} \mathcal{CC}^{(n)}.
\]
Let $\mathcal{QDC}$ denote the space of decorated cones $(C; \tilde{s})$ for $\tilde{s} \in \mathbb{Z}_{\leq 0}$. Extend $\Delta$ on $\mathcal{QCC}$ to $\mathcal{QDC}$ by derivation:

$$\Delta(C; \tilde{s}) = (\Delta \circ \delta_i)(C; \tilde{s} + e_i) = (D_i \circ \Delta)(C; \tilde{s} + e_i).$$

Then $\mathcal{QDC}$ is a connected coalgebra with derivations.
Regularized CZVs

- A meromorphic function $f(\vec{z})$ on $\mathbb{C}^k$ is said to have linear poles at zero if there are linear forms $L_i(\vec{z}) = \sum_j a_{ij}z_j$, such that $(\prod_i L_i)f$ is homomorphic at zero.

- Let $\mathcal{M}(\mathbb{C}^k)$ be the algebra of such functions and let $\mathcal{M}(\mathbb{C}^\infty) = \bigcup_k \mathcal{M}(\mathbb{C}^k)$.

- We also have the summation map

$$S : \mathbb{Q}C^C \to \mathcal{M}(\mathbb{C}^\infty), \quad S(C)(\vec{z}) := \sum_{\vec{n} \in C^0 \cap \mathbb{Z}^k} e^{-(\vec{z}, \vec{n})}.$$

- By taking derivations, $S$ extends to

$$S : \mathbb{Q}D\mathbb{C}^C \to \mathcal{M}(\mathbb{C}^\infty),

S(C; \vec{s}) := \zeta(C; \vec{s}; \vec{z}) := \sum_{\vec{n} \in C^0 \cap \mathbb{Z}^k} \frac{e^{-(\vec{z}, \vec{n})}}{n_1^{s_1} \cdots n_k^{s_k}}.$$

This can be regarded as a regularization of

$$\zeta(C; \vec{s}; 0) = \sum_{\vec{n} \in C^0 \cap \mathbb{Z}^k} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.$$
Algebraic Birkhoff Decomposition

- There is a linear decomposition

\[ \mathcal{M}(\mathbb{C}^\infty) = \mathcal{M}_+(\mathbb{C}^\infty) \oplus \mathcal{M}_-(\mathbb{C}^\infty) = \mathcal{M}_1(\mathbb{C}^\infty) \oplus \mathcal{M}_2(\mathbb{C}^\infty), \]

where \( \mathcal{M}_+(\mathbb{C}^\infty) = Hol(\mathbb{C}^\infty) \) is the space of functions holomorphic at 0 and \( \mathcal{M}_-(\mathbb{C}^\infty) \) is spanned by

\[ \sum \frac{h(\ell_1, \cdots, \ell_m)}{L_1^{r_1} \cdots L_n^{r_n}}, \]

where \( h \in \mathcal{M}_+(\mathbb{C}^\infty), \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \) independent linear forms such that \( (\ell_i, L_j) = 0, \forall i, j. \)

- Together with the coproduct on \( \mathbb{Q}_{DC} \), we obtain a (Birkhoff) decomposition

\[ S = S_1^{*(-1)} \star S_2, \]

where \( S_i : \mathbb{Q}_{DC} \rightarrow \mathcal{M}_i(\mathbb{C}^\infty). \)

- The value \( \zeta(C; \vec{s}) := S_1^{*(-1)}(S; \vec{s})(0) \) is called the renormalized conical zeta value of \( (C; \vec{s}). \)
The (classical) Euler-Maclaurin formula relates the discrete sum
\[ S(\varepsilon) := \sum_{k=0}^{\infty} e^{-\varepsilon k} = \frac{1}{1-e^{-\varepsilon}} \]
for positive \( \varepsilon \) to the integral
\[ I(\varepsilon) := \int_{0}^{\infty} e^{-\varepsilon x} \, dx = \frac{1}{\varepsilon} \]
by means of the interpolator
\[ \mu(\varepsilon) := S(\varepsilon) - I(\varepsilon) = S(\varepsilon) - \frac{1}{\varepsilon} = \frac{1}{2} + \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!} \varepsilon^{2k-1} + o(\varepsilon^{2K}) \quad \text{for all } K \in \mathbb{N} \]
which is holomorphic at \( \varepsilon = 0 \).

This formula becomes a special case of the Euler-Maclaurin formula for cone, of Berline and Vergne, when the cone is taken to be \([0, \infty)\).
Euler-Maclaurin Formula for Cones

- For a smooth cone $C$, define

$$I(C)(\vec{z}) := \int_C e^{-\langle \vec{x}, \vec{z} \rangle} d\vec{x}.$$ 

This gives rise to a map

$$I : \mathcal{Q}CC \to \mathcal{M}(\mathcal{C}^\infty).$$

- Euler-Maclaurin Formula (Berline-Vergne) There is a map (interpolator)

$$\mu : \mathcal{Q}CC \to Hol(\mathcal{C}^\infty),$$

such that

$$S(C) = \sum_{F \text{ face of } C} \mu(t(C, F)) I(F).$$
Note that $S_+$ and $S_-$ are unique such that $S_+(\ker \varepsilon) \subseteq M_+$ and $S_-(\ker \varepsilon) \subseteq M_-$ where $\varepsilon : \mathbb{Q}\text{DCC} \to \mathbb{Q}$ is the counit.

Thus comparing with $S = \mu \star I$ and $I : \mathbb{Q}\text{DCC} \to M_-$, we obtain

$$\mu = S_+^{*(-1)}, \quad I = S_-.$$

Further,

$$\mu = \pi_+ S.$$
References

Thank You!