Renormalization and Euler-Maclaurin Formula on Cones

Li GUO
(joint work with Sylvie Paycha and Bin Zhang)

Rutgers University at Newark

Outline

- Conical zeta values and multiple zeta values;
- Double shuffle relations and double subdivision relations;
- Renormalization of conical zeta values;
- Euler-Maclaurin formula.

Cones

- A (closed polyhedral) cone in $\mathbb{R}^k_{\geq 0}$ is defined to be the convex set

$$\langle v_1, \cdots, v_n \rangle := \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n, v_i \in \mathbb{R}^k_{\geq 0}, 1 \leq i \leq n.$$

- The interior of a cone $\langle v_1, \cdots, v_n \rangle$ is an open (polyhedral) cone

$$\langle v_1, \cdots, v_n \rangle^o := \mathbb{R}_{>0} v_1 + \cdots + \mathbb{R}_{>0} v_n.$$

- The set $\{v_1, \cdots, v_n\}$ is called the generating set or the spanning set of the cone. The dimension of a cone is the dimension of linear subspace generated by it.
- Let $\mathcal{C}_k$ (resp. $\mathcal{OC}_k$) denote the set of closed (resp. open cones) in $\mathbb{R}^k$, $k \geq 1$. For $k = 0$ we set $\mathcal{C}_0 = \{0\}$ (resp. $\mathcal{OC}_0 = \{0\}$) by convention. Through the natural inclusions $\mathcal{C}_k \to \mathcal{C}_{k+1}$ (resp. $\mathcal{OC}_k \to \mathcal{OC}_{k+1}$) from the natural inclusion $\mathbb{R}^k \to \mathbb{R}^{k+1}$, we define $\mathcal{C} = \varinjlim \mathcal{C}_k$ (resp. $\mathcal{OC} = \varinjlim \mathcal{OC}_k$).

- A simplicial cone is defined to be a cone spanned by linearly independent vectors.
- A rational cone is a cone spanned by vectors in $\mathbb{Z}^k \subseteq \mathbb{R}^k$.
- A smooth cone is a rational cone with a spanning set that is a part of a basis of $\mathbb{Z}^k \subseteq \mathbb{R}^k$. In this case, the spanning set is unique and is called the primary set of the cone.
- A cone is called strongly convex or pointed if it does not contain any linear subspace.
- A subdivision of a closed cone $C \in \mathcal{C}_k$ is a set $\{C_1, \cdots, C_r\} \subseteq \mathcal{C}_k$ such that $C = \cup_{i=1}^r C_i$, $C_1, \cdots, C_r$ have the same dimension C and intersect along their faces. The faces of the relative interior give an open subdivision of C^o:

$$\langle e_1, e_2 \rangle = \langle e_1, e_1 + e_2 \rangle \sqcup \langle e_1 + e_2, e_e \rangle$$

$$\Rightarrow \langle e_1, e_2 \rangle^o = \langle e_1, e_1 + e_2 \rangle^o \sqcup \langle e_1 + e_2, e_e \rangle^o \sqcup \langle e_1 + e_2 \rangle^o.$$

- For $\vec{x} = (x_1, \cdots, x_k)$ and $\vec{y} = (y_1, \cdots, y_k)$ in $\mathbb{R}^k$, let $(\vec{x}, \vec{y})$ denote the inner product $x_1 y_1 + \cdots + x_k y_k$. Through this inner product, $\mathbb{R}^k$ is identified with its own dual space $(\mathbb{R}^k)^*$.

Conical zeta values

- Let C be a smooth cone. The conical zeta function of C is
$$\zeta(C;\vec{s}) := \sum_{(n_1,\cdots,n_k)\in C^o\cap\mathbb{Z}^k} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}, \vec{s}\in\mathbb{C}^k,$$
if the sum converges. When $s_i, 1\le i\le k$, are integers, $\zeta(\vec{s})$ is called a conical zeta value (CZV). Convention: $0^s = 1$ for any s. Hence $\zeta(\vec{s})$ does not depend on the choice of k.
- If $s_i \ge 2, 1\le i\le k$, then $\zeta(C;\vec{s})$ converges.
- If $\{C_i\}_i$ is an open cone subdivision of C, then
$$\zeta(C;\vec{s}) = \sum_i \zeta(C_i;\vec{s}).$$
- The cone subdivision
$$\langle e_1, e_2\rangle^o = \langle e_1, e_1+e_2\rangle^o \sqcup \langle e_1+e_2, e_2\rangle^o \sqcup \langle e_1+e_2\rangle^o$$
gives
$$\zeta(\langle e_1,e_2\rangle^o;(s_1,s_2)) = \zeta(\langle e_1,e_1+e_2\rangle^o;(s_1,s_2)) \\ +\zeta(\langle e_1+e_2,e_2\rangle^o;(s_1,s_2) + \zeta(\langle e_1+e_2\rangle^o;(s_1,s_2).$$

Chen cones and multiple zeta values

- A Chen cone of dimension k is a cone

$$C_{k,\sigma} := \langle e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)}, \cdots, e_{\sigma(1)} + \cdots + e_{\sigma(k)} \rangle,$$

  where $\sigma \in S_k$. Let C_k denote the standard Chen cone spanned by $\{e_1, \cdots, e_k\}$.
- Then $\zeta(C_{k,\sigma}; s_1, \cdots, s_k) = \zeta(s_{\sigma(1)}, \cdots, s_{\sigma(k)})$,

$$\zeta(C_{k,\mathrm{id}}; s_1, \cdots, s_k) = \zeta(s_1, \cdots, s_k).$$

- The stuffle product of two MZVs $\zeta(r_1, \cdots, r_k)$ and $\zeta(s_1, \cdots, s_\ell)$ is recovered by the subdivision of the cone $C_k \times C_\ell$ (direct product) into Chen cones.
- For example, the open cone subdivision relation

$$\zeta(\langle e_1, e_2\rangle^o; (s_1, s_2)) = \zeta(\langle e_1, e_1 + e_2\rangle^o; (s_1, s_2)) \\ +\zeta(\langle e_1 + e_2, e_2\rangle^o; (s_1, s_2) + \zeta(\langle e_1 + e_2\rangle^o; (s_1, s_2)$$

  gives the stuffle relation

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).$$

Multiple zeta values

- The multiple zeta value algebra is

$$\mathbf{MZV} := \mathbb{Q}\{\zeta(s_1, \cdots, s_k) \mid s_i \geq 1, s_1 \geq 1\}.$$

- The quasi-shuffle algebra $\mathcal{H}^*$ has the underlying vector space

$$\mathbb{Q}\langle z_s \mid s \geq 1 \rangle$$

with the quasi-shuffle product. It contains the subalgebra

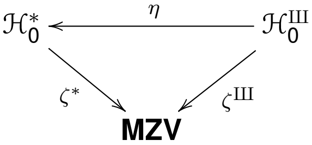
$$\mathcal{H}_0^* := \mathbb{Q}.1 \oplus \left(\bigoplus_{s_1 \geq 2} \mathbb{Q} z_{s_1} \cdots z_{s_k} \right) \subseteq \mathcal{H}^*.$$

The stuffle relation of MZVs is encoded in the algebra homomorphism

$$\zeta^* : \mathcal{H}_0^* \longrightarrow \mathbf{MZV}, \quad z_{s_1} \cdots z_{s_k} \mapsto \zeta(s_1, \cdots, s_k).$$

Double shuffle relation

- The shuffle algebra $\mathcal{H}^{ш}$ has the underlying vector space $\mathbb{Q}\langle x_0, x_1\rangle$ equipped with the shuffle product of words. It contains the subalgebra
$$\mathcal{H}_0^{ш} := \mathbb{Q}.1 \bigoplus x_0 \mathcal{H}^{ш} x_1 .$$
The shuffle relation of the MZVs is encoded in the algebra homomorphism
$$\zeta^{ш} : \mathcal{H}_0^{ш} \to \textbf{MZV}, \quad x_0^{s_1-1}x_1 \cdots x_0^{s_k-1}x_1 \mapsto \zeta(s_1, \cdots, s_k).$$
- There is a natural bijection of abelian groups (but *not* algebras)
$$\eta : \mathcal{H}_0^{ш} \to \mathcal{H}_0^{*}, \quad 1 \leftrightarrow 1,\ x_0^{s_1-1}x_1 \cdots x_0^{s_k-1}x_1 \leftrightarrow z_{s_1} \cdots z_{s_k}.$$
- Then the fact that MZVs can be multiplied in two ways is reflected by



Double shuffle relation
$$\zeta^{*}\big(w_1 * w_2 - \eta(\eta^{-1}(w_1) ш \eta^{-1}(w_2))\big), \quad w_1, w_2 \in \mathcal{H}_0^{*}.$$

Linearly constrained zeta values (LCZ)

- Let $\langle v_1, \cdots, v_k\rangle$ be a smooth close cone with ita (unique) primitive generating set.
- For $s_1, \cdots, s_k \geq 1$, called the formal expression $[v_1]^{s_1}\cdots[v_k]^{s_k}$ a decorated smooth cone.
- Define the linearly constrained zeta value (LZV)

$$\zeta^c([v_1]^{s_1}\cdots[v_k]^{s_k}) := \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{1}{(a_{11}m_1+\cdots+a_{1r}m_r)^{s_1}\cdots(a_{k1}m_1+\cdots+a_{kr}m_r)^{s_k}}$$

if the sum is convergent, where $v_i=\sum_{j=1}^{r}a_{ij}e_j, 1\leq i\leq k$. When $[v_1]\cdots[v_k]$ is a Chen cone $[e_1]\cdots[e_1+\cdots+e_k]$, then we have

$$\zeta^c([v_1]^{s_1}\cdots[v_k]^{s_k})=\zeta(s_1,\cdots,s_k).$$

Subdivision of decorated closed cones

- Let $\{\langle v_{i1}, \cdots, v_{ik}\rangle\}_i$ be a smooth subdivision of the smooth cone $\langle v_1, \cdots, v_k\rangle$. Call $\sum_i [v_{i1}] \cdots [v_{ik}]$ an algebraic subdivision of $[v_1] \cdots [v_k]$.
- Let $[v_1]^{s_1} \cdots [v_k]^{s_k}$ be a decorated smooth closed cone.
- Define $\delta_{e_i}([v_1]^{s_1} \cdots [v_k]^{s_k}) = \sum_j s_j(e_i, v_j)[v_1]^{s_1} \cdots [v_j]^{s_j+1} \cdots [v_k]^{s_k}$. For $u = \sum_i c_i e_i$, define $\delta_u = \sum_i c_i \delta_{e_i}$. Then $[v_1]^{s_1} \cdots [v_k]^{s_k} = \frac{1}{(s_1-1)!\cdots(s_k-1)!}\delta_{v_1^*}^{s_1-1} \cdots \delta_{v_k^*}^{s_k-1}([v_{i1}] \cdots [v_{ik}])$.
- Call
$$\sum_i \frac{1}{(s_1-1)!\cdots(s_k-1)!}\delta_{v_1^*}^{s_1-1} \cdots \delta_{v_k^*}^{s_k-1}([v_{i1}] \cdots [v_{ik}])$$
an algebraic subdivision of $[v_1]^{s_1} \cdots [v_k]^{s_k}$. Here $v_1^*, \cdots, v_k^*$ is a dual basis of $v_1, \cdots, v_k$.
- Let $D = \sum_i a_i D_i$ be an algebraic subdivision of a decorated smooth cone D. Then
$$\zeta^c(D) = \sum_i a_i \zeta^c(D_i).$$
- This generalizes the shuffle relation of MZVs.

Relating open and closed subdivisions

- Let $GL_r(\mathbb{Z})$ denote the set of $r \times r$ unimodular matrices. Let $M \in GL_r(\mathbb{Z})$ and $\vec{s} := (s_1, \ldots, s_r) \in \mathbb{Z}^r_{\geq 0}$. Let $v_1, \cdots, v_r$ and $u_1, \cdots, u_r$ be the row and column vectors of M. The (decorated) cone pair associated with M and $\vec{s}$ is the pair (C, D) consisting of the decorated open cone $C := C_{M,\vec{s}} = (\langle u_1, \cdots, u_r \rangle^o, \vec{s})$ and the decorated closed cone $D := D_{M,\vec{s}} = [v_1]^{s_1} \cdots [v_r]^{s_r}$. We call the pair convergent if the corresponding ζ-values $\zeta^0(C)$ and $\zeta^c(D)$ converge.
- Let $\mathcal{DTP}$ denote the set of cone pairs $(C_{M,\vec{s}}, D_{M,\vec{s}})$ where $M \in O(\mathbb{Z})$ and $\vec{s} \in \mathbb{Z}^r_{\geq 0}$. Let

$$p^o : \mathbb{Q}\mathcal{DTP} \to \mathbb{Q}\mathcal{DC}$$

and

$$p^c : \mathbb{Q}\mathcal{DTP} \to \mathbb{Q}\mathcal{DMC}$$

denote the natural projections.
- For any cone pair $(C, D) \in \mathcal{DTP}$, we have

$$\zeta^o(C) = \zeta^c(D),$$

if either side makes sense.

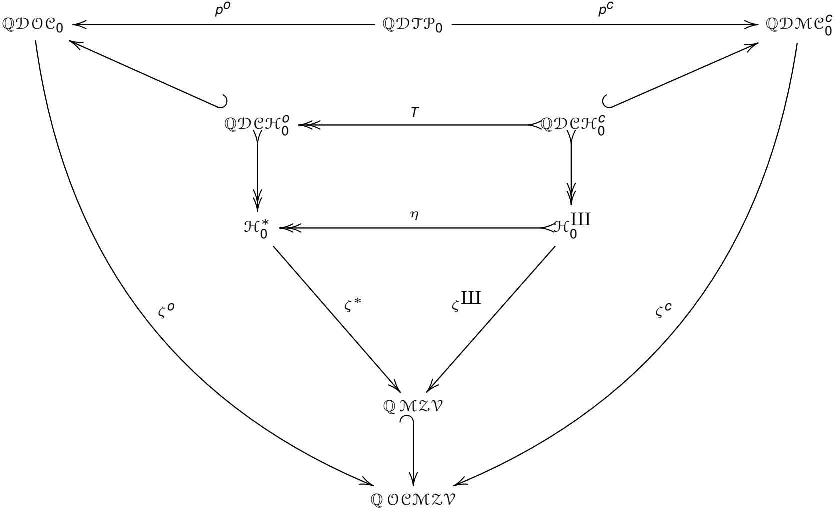
Double subdivision relation

- Let (C, D) be a convergent cone pair. Let $\{C_i\}_i$ be an open subdivision of the decorated open cone C and let $\sum_j c_j D_j$ be a subdivision of the decorated closed cone D. Also let $D_j^T \in \mathcal{DC}$ be the transpose cone of D^j, that is, (D_j^T, D_j) is a cone pair. Then

$$\sum_i C_i - \sum_j c_j D_j^T \qquad (1)$$

  lies in the kernel of ζ^o. It is called a double subdivision relation.
- For any not necessarily convergent cone pair (C, D), let $\{C_i\}$ be a subdivision of C and $\sum_j a_j D_j$ a subdivision of D. If $\sum_i C_i - \sum_j a_j D_j^T$ is in $\mathbb{Q}\mathcal{DC}$, then it is called an extended double subdivision relation.
- Hunch. The kernel of ζ^o is the subspace I_{EDS} of $\mathbb{Q}\mathcal{DC}$ generated by the extended double subdivision relations.

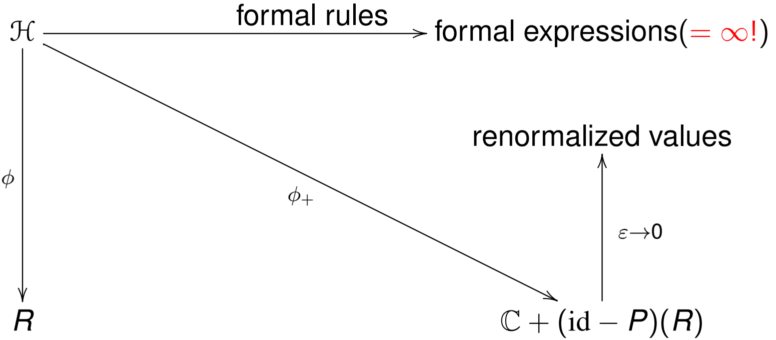
Double subdivision relation

- 

Algebraic Birkhoff Decomposition

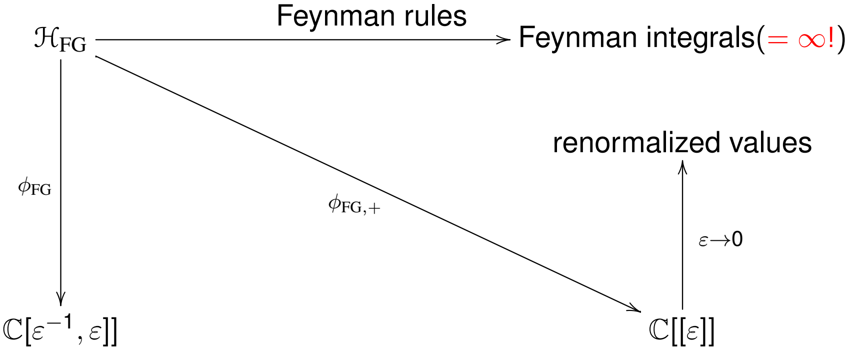
- Algebraic Birkhoff Decomposition. Let $\mathcal{H}$ be a connected filtered Hopf algebra, $R = P(R) \oplus (\mathrm{id} - P)(R)$ a commutative Rota-Baxter algebra with an idempotent Rota-Baxter operator P. Any algebra homomorphism $\phi : \mathcal{H} \to R$ has a unique decomposition into algebra homomorphisms

$$\phi = \phi_-^{-1} \star \phi_+, \qquad \begin{cases} \phi_- : \mathcal{H} \to \mathbb{C} + P(R) \text{ (counter term)} \\ \phi_+ : \mathcal{H} \to \mathbb{C} + (\mathrm{id} - P)(R) \text{ (renormalization)} \end{cases}$$



- In QFT renormalization (Dim-Reg scheme), we take the triple $(\mathcal{H}_{\mathrm{FG}}, R_{\mathrm{FG}}, \phi_{\mathrm{FG}})$ with
- Hopf algebra $\mathcal{H}_{\mathrm{FG}}$ of Feynman graphs;
- $R_{\mathrm{FG}} = \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$ of Laurent series, with the pole part projection P;
- $\phi_{\mathrm{FG}} : \mathcal{H}_{\mathrm{FG}} \to R_{\mathrm{FG}}$ from dimensional regularized Feynman rule.
- Then Algebraic Birkhoff Decomposition gives

$$\phi_{\mathrm{FG}} = \phi_{\mathrm{FG},-} \star \phi_{\mathrm{FG},+}$$



Generalized Algebraic Birkhoff Decomposition

- Let $\mathbf{C} = \bigoplus_{n\geq 0} \mathbf{C}^{(n)}$ be a (co)differential connected coalgebra (so $\mathbf{C}^{(0)} = \mathbf{k}J$) with counit $\varepsilon : \mathbf{C} \to \mathbf{k}$ and coderivations $\delta_\sigma, \sigma \in \Sigma$. Let A be a differential algebra with derivations $\partial_\sigma, \sigma \in \Sigma$. Let $A = A_1 \oplus A_2$ be a linear decomposition such that $1_A \in A_1$ and

$$\partial_\sigma(A_i) \subseteq A_i, \quad i = 1, 2, \quad \sigma \in \Sigma.$$

Let P be the projection of A to A_1 along A_2. Denote

$$\mathcal{G}(\mathbf{C}, A) := \{\phi : \mathbf{C} \to A \,|\, \phi(J) = 1_A, \partial_\sigma \phi = \phi \delta_\sigma, \sigma \in \Sigma\}.$$

Then any $\phi \in \mathcal{G}(\mathbf{C}, A)$ has a unique decomposition

$$\varphi = \varphi_1^{*(-1)} * \varphi_2,$$

where $\varphi_i \in \mathcal{G}(\mathbf{C}, A), i = 1, 2$, satisfy $(\ker \varepsilon) \subseteq A_i$ (hence $\varphi_i : \mathbf{C} \to \mathbf{k}1_A + A_i$). If moreover A_1 is a subalgebra of A then $\phi_1^{*(-1)}$ lies in $\mathcal{G}(\mathbf{C}, A_1)$.

Transverse cones

- Identify $V_k := \mathbb{R}^k$ with its dual through a fixed inner product $(\cdot,\cdot)$.
- For a cone C, let $\mathrm{lin}(C)$ denote the subspace spanned by C.
- For any closed cone C and its face F, define the transverse cone (Berline and Vergne) $t(C,F)$ along F to be the projection of C to $F^\perp$, where $F^\perp = \mathrm{lin}_C^\perp(F)$ is the orthogonal completion of $\mathrm{lin}(F)$ in $\mathrm{lin}(C)$.
- For example, the transverse cone of $\langle e_1, e_1 + e_2\rangle$ along $\langle e_1 + e_2\rangle$ is $\langle e_1 - e_2\rangle$.

Coproduct of cones

- We equip the linear space $\mathbb{Q}\mathcal{C}\mathcal{C}$ of close cones with a coproduct

$$\Delta : \mathbb{Q}\mathcal{C}\mathcal{C} \to \mathbb{Q}\mathcal{C}\mathcal{C} \otimes \mathbb{Q}\mathcal{C}\mathcal{C}, \Delta C := \sum_{F \preceq C} t(C, F) \otimes F$$

and a counit

$$\varepsilon : \mathbb{Q}\mathcal{C}\mathcal{C} \to \mathbb{Q}, \varepsilon(C) = \begin{cases} 1, & C = \{0\}, \\ 0, & C \neq \{0\}. \end{cases}$$

- With $\mathcal{C}\mathcal{C}^{(n)} := \{C \in \mathcal{C}\mathcal{C} | \dim C = n\}, n \geq 0$, we have a connected coalgebra

$$\mathcal{C}\mathcal{C} = \oplus_{n \geq 0} \mathcal{C}\mathcal{C}^{(n)}.$$

Decorated closed cones

- Let $\mathbb{QDC}$ denote the space of decorated cones $(C; \vec{s})$ for $\vec{s} \in \mathbb{Z}_{\leq 0}$. Extend Δ on $\mathbb{QCC}$ to $\mathbb{QDC}$ by derivation:

$$\Delta(C; \vec{s}) = (\Delta \circ \delta_i)(C; \vec{s} + e_i) = (D_i \circ \Delta)(C; \vec{s} + e_i).$$

- Then $\mathbb{QDC}$ is a connected coalgebra with derivations.

Regularized CZVs

- A meromorphic function $f(\vec{z})$ on $\mathbb{C}^k$ is said to have linear poles at zero if there are linear forms $L_i(\vec{z}) = \sum_j a_{ij} z_j$, such that $(\prod_i L_i) f$ is homomorphic at zero.
- Let $\mathcal{M}(\mathbb{C}^k)$ be the algebra of such functions and let $\mathcal{M}(\mathbb{C}^\infty) = \cup_k \mathcal{M}(\mathbb{C}^k)$.
- We also have the summation map

$$S : \mathbb{Q}\mathcal{C}\mathcal{C} \to \mathcal{M}(\mathbb{C}^\infty), S(C)(\vec{z}) := \sum_{\vec{n} \in C^o \cap \mathbb{Z}^k} e^{-(\vec{z}, \vec{n})}.$$

- By taking derivations, S extends to

$$S : \mathbb{Q}\mathcal{D}\mathcal{C}\mathcal{C} \to \mathcal{M}(\mathbb{C}^\infty),$$

$$S(C; \vec{s}) := \zeta(C; \vec{s}; \vec{z}) := \sum_{\vec{n} \in C^o \cap \mathbb{Z}^k} \frac{e^{-(\vec{z}, \vec{n})}}{n_1^{s_1} \cdots n_k^{s_k}}.$$

  This can be regarded as a regularization of

$$\zeta(C; \vec{s}; 0) = \sum_{\vec{n} \in C^o \cap \mathbb{Z}^k} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.$$

Algebraic Birkhoff Decomposition

- There is a linear decomposition

$$\mathcal{M}(\mathbb{C}^\infty) = \mathcal{M}_+(\mathbb{C}^\infty) \oplus \mathcal{M}_-(\mathbb{C}^\infty) = \mathcal{M}_1(\mathbb{C}^\infty) \oplus \mathcal{M}_2(\mathbb{C}^\infty),$$

  where $\mathcal{M}_+(\mathbb{C}^\infty) = Hol(\mathbb{C}^\infty)$ is the space of functions holomorphic at 0 and $\mathcal{M}_-(\mathbb{C}^\infty)$ is spanned by

$$\sum \frac{h(\ell_1, \cdots, \ell_m)}{L_1^{r_1} \cdots L_n^{r_n}},$$

  where $h \in \mathcal{M}_+(\mathbb{C}^\infty)$, $\ell_1, \cdots, \ell_m, L_1, \cdots, L_n$ independent linear forms such that $(\ell_i, L_j) = 0, \forall i, j$.

- Together with the coproduct on $\mathbb{Q}\mathcal{D}\mathcal{C}$, we obtain a (Birkhoff) decomposition

$$S = S_1^{\star(-1)} \star S_2,$$

  where $S_i : \mathbb{Q}\mathcal{D}\mathcal{C} \longrightarrow \mathcal{M}_i(\mathbb{C}^\infty)$.

- The value $\zeta(C; \vec{s}) := S_1^{\star(-1)}(S; \vec{s})(0)$ is called the renormalized conical zeta value of $(C; \vec{s})$.

Classical Euler-Maclaurin Formula

- The (classical) Euler-Maclaurin formula relates the discrete sum $S(\varepsilon) := \sum_{k=0}^{\infty} e^{-\varepsilon k} = \frac{1}{1-e^{-\varepsilon}}$ for positive ε to the integral

$$I(\varepsilon) := \int_0^{\infty} e^{-\varepsilon x}\, dx = \frac{1}{\varepsilon}$$

  by means of the interpolator

$$\mu(\varepsilon) := S(\varepsilon) - I(\varepsilon) = S(\varepsilon) - \frac{1}{\varepsilon} = \frac{1}{2} + \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!}\varepsilon^{2k-1} + o(\varepsilon^{2K}) \quad \text{for all } K \in$$

  which is holomorphic at $\varepsilon = 0$.
- This formula becomes a special case of the Euler-Maclaurin formula for cone, of Berline and Vergne, when the cone is taken to be $[0, \infty)$.

Euler-Maclaurin Formula for Cones

- For a smooth cone C, define

$$I(C)(\vec{z}) := \int_C e^{-(\vec{x},\vec{z})} d\vec{x}.$$

This gives rise to a map

$$I : \mathbb{Q}\mathcal{C}\mathcal{C} \to \mathcal{M}(\mathbb{C}^\infty).$$

- Euler-Maclaurin Formula (Berline-Vergne) There is a map (interpolator)

$$\mu : \mathbb{Q}\mathcal{C}\mathcal{C} \to Hol(\mathbb{C}^\infty),$$

such that

$$S(C) = \sum_{F \text{ face of } C} \mu(t(C,F))I(F).$$

Birkhoff Factorization and Euler-Maclaurin

- Note that S_+ and S_- are unique such that $S_+(\ker\varepsilon) \subseteq \mathcal{M}_+$ and $S_-(\ker\varepsilon) \subseteq \mathcal{M}_-$ where $\varepsilon : \mathbb{Q}\mathfrak{D}\mathcal{C}\mathcal{C} \to \mathbb{Q}$ is the counit.
- Thus comparing with $S = \mu \star I$ and $I : \mathbb{Q}\mathfrak{D}\mathcal{C}\mathcal{C} \to \mathcal{M}_-$, we obtain

$$\mu = S_+^{\star(-1)}, \qquad I = S_-.$$

  Further,

$$\mu = \pi_+ S.$$

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- **Thank You!**