Loop of formal diffeomorphisms

A. Frabetti (Lyon, France)

based on a work in progress with

Ivan P. Shestakov (Sao Paulo, Brasil)

Potsdam, 8–12 February, 2016

Motivation: from renormalization Hopf algebras to series

- Diffeomorphism groups fix structure in geometry and physics.

Motivation: from renormalization Hopf algebras to series

- Diffeomorphism groups fix structure in geometry and physics.
- Taylor expansion gives **formal diffeomorphisms**, ok for perturbations. They form **proalgebraic groups**, represented by commutative Hopf algebras on infinitely many generators.

Motivation: from renormalization Hopf algebras to series

- Diffeomorphism groups fix structure in geometry and physics.
- Taylor expansion gives **formal diffeomorphisms**, ok for perturbations. They form **proalgebraic groups**, represented by commutative Hopf algebras on infinitely many generators.
- In pQFT, ren. Hopf algebras do represent groups of **formal series on the coupling constants** [Connes-Kreimer 1998, Pinter 2001, Keller 2010]

Motivation: from renormalization Hopf algebras to series

- Diffeomorphism groups fix structure in geometry and physics.
- Taylor expansion gives **formal diffeomorphisms**, ok for perturbations. They form **proalgebraic groups**, represented by commutative Hopf algebras on infinitely many generators.
- In pQFT, ren. Hopf algebras do represent groups of **formal series on the coupling constants** [Connes-Kreimer 1998, Pinter 2001, Keller 2010]
- Renormalization Hopf algebras:
  - are **right-sided combinatorial Hopf alg** [Loday-Ronco 2008, Brouder-AF-Menous 2011]
  - are all related to **operads** and produce $\mathcal{P}$-expanded series [Chapoton 2003, van der Laan 2003, AF 2008]
  - admit **non-commutative lifts** [Brouder-AF 2000, 2006, Foissy 2001]

Motivation: from renormalization Hopf algebras to series

- Diffeomorphism groups fix structure in geometry and physics.
- Taylor expansion gives **formal diffeomorphisms**, ok for perturbations. They form **proalgebraic groups**, represented by commutative Hopf algebras on infinitely many generators.
- In pQFT, ren. Hopf algebras do represent groups of **formal series on the coupling constants** [Connes-Kreimer 1998, Pinter 2001, Keller 2010]
- Renormalization Hopf algebras:
  - are **right-sided combinatorial Hopf alg** [Loday-Ronco 2008, Brouder-AF-Menous 2011]
  - are all related to **operads** and produce $\mathcal{P}$-expanded series [Chapoton 2003, van der Laan 2003, AF 2008]
  - admit **non-commutative lifts** [Brouder-AF 2000, 2006, Foissy 2001]
- **Puzzling situation:**
  - Series with coefficients in a non-comm. algebra A do appear in physics, but their commutative representative Hopf algebra is not functorial in A cf. [Van Suijlekom 2007] for QED.
  - These series are related to some non-commutative Hopf algebras which are functorial in A.

Aim of the talk

How are series related to non-commutative Hopf algebras?

Aim of the talk

How are series related to non-commutative Hopf algebras?

- Toy model: the set of formal diffeomorphisms in one variable

$$\mathrm{Diff}(A) = \Big\{ a(\lambda) = \lambda + \sum_{n\geqslant 1} a_n \; \lambda^{n+1} \mid a_n \in A \Big\}$$

with composition law $(a \circ b)(\lambda) = a\big(b(\lambda)\big)$ and unit $e(\lambda) = \lambda$, when A is a unital associative algebra, but **not commutative**.

Aim of the talk

How are series related to non-commutative Hopf algebras?

- Toy model: the set of formal diffeomorphisms in one variable

$$\mathrm{Diff}(A) = \Big\{ a(\lambda) = \lambda + \sum_{n \geqslant 1} a_n \, \lambda^{n+1} \mid a_n \in A \Big\}$$

with composition law $(a \circ b)(\lambda) = a\big(b(\lambda)\big)$ and unit $e(\lambda) = \lambda$, when A is a unital associative algebra, but **not commutative**.

Examples of non-commutative coefficients A:

$M_4(\mathbb{C})$ matrix algebra
cf. QED renormalization [Brouder-AF-Krattenthaler 2001, 2006]

$T(E)$ tensor algebra
cf. renormalization functor [Brouder-Schmitt 2002]
and work in progress on bundles with Brouder and Dang

$\mathcal{L}(\mathcal{H})$ linear operators on a Hilbert space

Aim of the talk

How are series related to non-commutative Hopf algebras?

- Toy model: the set of formal diffeomorphisms in one variable

$$\mathrm{Diff}(A) = \Big\{ a(\lambda) = \lambda + \sum_{n\geqslant 1} a_n\ \lambda^{n+1} \mid a_n \in A \Big\}$$

  with composition law $(a \circ b)(\lambda) = a\big(b(\lambda)\big)$ and unit $e(\lambda) = \lambda$, when A is a unital associative algebra, but **not commutative**.

  Examples of non-commutative coefficients A:

  $M_4(\mathbb{C})$ matrix algebra
  cf. QED renormalization [Brouder-AF-Krattenthaler 2001, 2006]

  $T(E)$ tensor algebra
  cf. renormalization functor [Brouder-Schmitt 2002]
  and work in progress on bundles with Brouder and Dang

  $\mathcal{L}(\mathcal{H})$ linear operators on a Hilbert space

- Two problems:
  1) define (pro)algebraic groups on **non-commutative algebras**
  2) modify because $\mathrm{Diff}(A)$ is **not a group**!

Lie and (pro)algebraic groups on commutative algebras

Group		**Function algebra**
G Lie group or (pro)algebraic $G(A) \cong \operatorname{Hom}_{uCom}(R[G], A)$ convolution group	representations → ← algebraic group	$R[G] = \mathcal{O}(G)$ Hopf com dense in $C^\infty(G)$

Lie and (pro)algebraic groups on commutative algebras

Group

G Lie group
or (pro)algebraic

$$G(A) \cong \operatorname*{Hom}_{uCom}(R[G], A)$$
convolution group

representations →

← algebraic group

Function algebra

$$R[G] = \mathcal{O}(G)$$
Hopf com

dense in $C^\infty(G)$

infinitesimal structure ↓

distributions supported at e ↘

↖ group-like elements

↕ Hopf algebra duality

Lie algebra

$$\mathfrak{g} = T_e G \cong \operatorname{Prim} U\mathfrak{g}$$

$$\mathfrak{g}_A = \mathfrak{g} \otimes A$$

$$uAs \to Lie : A \mapsto A_L$$
$$[a, b] = ab - ba$$

algebra ext. →

← primitives

↔ adjoint functors

Enveloping algebra

$$U\mathfrak{g} \cong R[G]^*$$
Hopf cocom (not com)

$$U\mathfrak{g}_A \cong U\mathfrak{g} \otimes A$$

$$\operatorname*{Hom}_{Lie}(\mathfrak{g}, A_L) \cong \operatorname*{Hom}_{uAs}(U\mathfrak{g}, A)$$

Details on convolution groups and functorial Lie algebras

Let A be a **commutative** algebra and H be a **commutative Hopf** algebra.
Denote: multiplication m, unit u, coproduct Δ, counit ε and antipode S.

Details on convolution groups and functorial Lie algebras

Let A be a **commutative** algebra and H be a **commutative Hopf** algebra.
Denote: multiplication m, unit u, coproduct Δ, counit ε and antipode S.

- The set $\underset{uCom}{\mathrm{Hom}}(H, A)$ forms a group with

  convolution $\alpha * \beta = m_A\ (\alpha \otimes \beta)\ \Delta_H$

  unit $e = u_A\ \varepsilon_H$

  inverse $\alpha^{-1} = \alpha\ S_H$

Details on convolution groups and functorial Lie algebras

Let A be a **commutative** algebra and H be a **commutative Hopf** algebra.
Denote: multiplication m, unit u, coproduct Δ, counit ε and antipode S.

- The set $\underset{uCom}{\mathrm{Hom}}(H, A)$ forms a group with

$$\text{convolution} \qquad \alpha * \beta = m_A \; (\alpha \otimes \beta) \; \Delta_H$$

$$\text{unit} \qquad e = u_A \; \varepsilon_H$$

$$\text{inverse} \qquad \alpha^{-1} = \alpha \; S_H$$

- Let G be a (pro)algebraic group represented by the Hopf algebra $R[G]$, and let $x_1, x_2, \dots$ be generators of $R[G]$ (coordinate functions on G). Then the isomorphism $G(A) \cong \underset{uCom}{\mathrm{Hom}}(R[G], A)$ is given by

$$g \longmapsto \alpha_g : R[G] \to A$$

$$x_n \mapsto \alpha_g(x_n) = x_n(g).$$

Details on convolution groups and functorial Lie algebras

Let A be a **commutative** algebra and H be a **commutative Hopf** algebra.
Denote: multiplication m, unit u, coproduct Δ, counit ε and antipode S.

- The set $\operatorname{Hom}_{uCom}(H, A)$ forms a group with

$$\text{convolution} \qquad \alpha * \beta = m_A \, (\alpha \otimes \beta) \, \Delta_H$$

$$\text{unit} \qquad e = u_A \, \varepsilon_H$$

$$\text{inverse} \qquad \alpha^{-1} = \alpha \, S_H$$

- Let G be a (pro)algebraic group represented by the Hopf algebra $R[G]$, and let $x_1, x_2, \ldots$ be generators of $R[G]$ (coordinate functions on G). Then the isomorphism $G(A) \cong \operatorname{Hom}_{uCom}(R[G], A)$ is given by

$$g \longmapsto \alpha_g : R[G] \to A$$

$$x_n \mapsto \alpha_g(x_n) = x_n(g).$$

- Let $\mathfrak{g}$ be a Lie algebra with bracket $[\ ,\ ]$. Then $\mathfrak{g}_A = \mathfrak{g} \otimes A$ is also a Lie algebra with bracket

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab,$$

and $U\mathfrak{g}_A \cong U\mathfrak{g} \otimes A$.

Convolution groups on non-commutative algebras

Let A and H be unital **associative** algebras (not nec. commutative).

Convolution groups on non-commutative algebras

Let A and H be unital **associative** algebras (not nec. commutative).

- Even if H is a Hopf algebra, the convolution $\alpha * \beta = m_A\,(\alpha \otimes \beta)\,\Delta_H$ is not well defined on $\underset{uAs}{\mathrm{Hom}}(H, A)$, because it is not an algebra morphism.

Convolution groups on non-commutative algebras

Let A and H be unital **associative** algebras (not nec. commutative).

- Even if H is a Hopf algebra, the convolution $\alpha * \beta = m_A\ (\alpha \otimes \beta)\ \Delta_H$ is not well defined on $\underset{uAs}{\mathrm{Hom}}(H, A)$, because it is not an algebra morphism.
- Solve requiring a modified coproduct $\Delta^{\circledast} : H \to H \circledast H$, where $A \circledast B$ is the **free product** algebra with **concatenation** $a \otimes b \otimes a' \otimes b' \otimes \cdots$ instead of $(aa' \cdots) \otimes (bb' \cdots)$ as in $A \otimes B$.

  Then $m_A : A \otimes A \to A$ induces an algebra morphism $m_A^{\circledast} : A \circledast A \to A$, can define the convolution

  $$\alpha * \beta = m_A^{\circledast}\ (\alpha \circledast \beta)\ \Delta_H^{\circledast}$$

  and get a group [Zhang 1991, Bergman-Hausknecht 1996].

Convolution groups on non-commutative algebras

Let A and H be unital **associative** algebras (not nec. commutative).

- Even if H is a Hopf algebra, the convolution $\alpha * \beta = m_A\ (\alpha \otimes \beta)\ \Delta_H$ is not well defined on $\underset{uAs}{\mathrm{Hom}}(H, A)$, because it is not an algebra morphism.
- Solve requiring a modified coproduct $\Delta^{\circledast} : H \to H \circledast H$, where $A \circledast B$ is the **free product** algebra with **concatenation** $a \otimes b \otimes a' \otimes b' \otimes \cdots$ instead of $(aa' \cdots) \otimes (bb' \cdots)$ as in $A \otimes B$.

  Then $m_A : A \otimes A \to A$ induces an algebra morphism $m_A^{\circledast} : A \circledast A \to A$, can define the convolution

  $$\alpha * \beta = m_A^{\circledast}\ (\alpha \circledast \beta)\ \Delta_H^{\circledast}$$

  and get a group [Zhang 1991, Bergman-Hausknecht 1996].
- If $(H, \Delta^{\circledast})$ is a modified Hopf algebra, then the natural projection $\Delta = \pi \Delta^{\circledast} : H \to H \circledast H \to H \otimes H$ defines a **usual Hopf algebra!**

  This explains how invertible series $G(A) = \left\{ a(\lambda) = 1 + \sum a_n \lambda^n \right\}$ with $(a\dot{}b)(\lambda) = a(\lambda) b(\lambda)$ still form a proalgebraic group, represented by the algebra of **non commutative symmetric functions**

  $$H = \mathbb{K}\langle x_1, x_2, \ldots \rangle, \qquad \Delta^{\circledast}(x_n) = \sum x_m \otimes x_{n-m}.$$

  [Brouder-AF-Krattenthaler 2006, cf. AF-Manchon 2014]

Lie algebras with non-commutative coefficients

Let $\mathfrak{g}$ be a **Lie algebra** and A a unital **associative** algebra.

Lie algebras with non-commutative coefficients

Let $\mathfrak{g}$ be a **Lie algebra** and A a unital **associative** algebra.

- On the vector space $\mathfrak{g} \otimes A$ the simple law $[x \otimes a, y \otimes b] = [x, y] \otimes ab$ does not define a Lie bracket (Jacobi fails).

Lie algebras with non-commutative coefficients

Let $\mathfrak{g}$ be a **Lie algebra** and A a unital **associative** algebra.

- On the vector space $\mathfrak{g} \otimes A$ the simple law $[x \otimes a, y \otimes b] = [x, y] \otimes ab$ does not define a Lie bracket (Jacobi fails).
- In fact it is an **open problem** to define a Lie bracket on $\mathfrak{g} \otimes A$!

  Only known example is $\mathfrak{sl}_2 \otimes J$ where J is a Jordan algebra (commutative but not associative), but does not fit.

Lie algebras with non-commutative coefficients

Let $\mathfrak{g}$ be a **Lie algebra** and A a unital **associative** algebra.

- On the vector space $\mathfrak{g} \otimes A$ the simple law $[x \otimes a, y \otimes b] = [x, y] \otimes ab$ does not define a Lie bracket (Jacobi fails).
- In fact it is an **open problem** to define a Lie bracket on $\mathfrak{g} \otimes A$!

  Only known example is $\mathfrak{sl}_2 \otimes J$ where J is a Jordan algebra (commutative but not associative), but does not fit.
- Let us turn the problem: **what is the infinitesimal structure of a (pro)algebraic group $G(A)$ if A is not commutative?**

Lie algebras with non-commutative coefficients

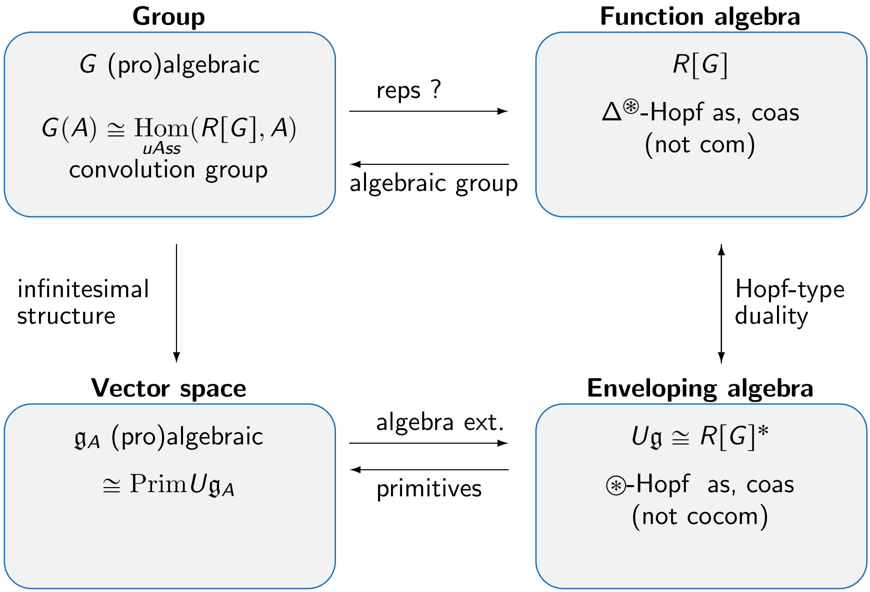
Let $\mathfrak{g}$ be a **Lie algebra** and A a unital **associative** algebra.

- On the vector space $\mathfrak{g} \otimes A$ the simple law $[x \otimes a, y \otimes b] = [x, y] \otimes ab$ does not define a Lie bracket (Jacobi fails).
- In fact it is an **open problem** to define a Lie bracket on $\mathfrak{g} \otimes A$!

  Only known example is $\mathfrak{sl}_2 \otimes J$ where J is a Jordan algebra (commutative but not associative), but does not fit.
- Let us turn the problem: **what is the infinitesimal structure of a (pro)algebraic group $G(A)$ if A is not commutative?**

  Hints come from **good triples of operads** [Loday 2008], if we apply functors to non-commutative algebras get the triple $(As, As, Vect)$: **$\mathfrak{g}_A$ is just a vector space!**

(Pro)algebraic groups on non-commutative algebras



Still a problem with diffeomorphisms!

- If A is a unital **associative** algebra (**not commutative**), the set

$$\mathrm{Diff}(A) = \Big\{ a(\lambda) = \lambda + \sum a_n \, \lambda^{n+1} \mid a_n \in A \Big\}$$

does not form a group because the composition is not associative:

$$\Big(a \circ (b \circ c) \Big)(\lambda) - \Big((a \circ b) \circ c \Big)(\lambda) = (a_1 b_1 c_1 - a_1 c_1 b_1) \, \lambda^4 + O(\lambda^5) \neq 0.$$

Still a problem with diffeomorphisms!

- If A is a unital **associative** algebra (**not commutative**), the set

$$\mathrm{Diff}(A) = \left\{ a(\lambda) = \lambda + \sum a_n \, \lambda^{n+1} \mid a_n \in A \right\}$$

does not form a group because the composition is not associative:

$$\Big(a \circ (b \circ c) \Big)(\lambda) - \Big((a \circ b) \circ c \Big)(\lambda) = (a_1 b_1 c_1 - a_1 c_1 b_1) \, \lambda^4 + O(\lambda^5) \neq 0.$$

- However the Faà di Bruno Hopf algebra $H_{\mathrm{FdB}} = R[\mathrm{Diff}]$ lifts up to a non commutative Hopf algebra $H^{\mathrm{nc}}_{\mathrm{FdB}} = \mathbb{K}\langle x_1, x_2, ... \rangle$ with

$$\Delta^{\mathrm{nc}}_{\mathrm{FdB}}(x_n) = \sum_{m=0}^{n} x_m \otimes \sum_{(k)} x_{k_0} \cdots x_{k_m} \qquad (x_0 = 1),$$

where $(k) = (k_0, k_1, ..., k_m)$ with $k_i \geqslant 0$ and $k_0 + k_1 + \cdots + k_m = n - m$ [Brouder-AF-Krattenthaler 2006].

Still a problem with diffeomorphisms!

- If A is a unital **associative** algebra (**not commutative**), the set

$$\mathrm{Diff}(A) = \left\{ a(\lambda) = \lambda + \sum a_n\ \lambda^{n+1} \mid a_n \in A \right\}$$

does not form a group because the composition is not associative:

$$\Big(a \circ (b \circ c)\Big)(\lambda) - \Big((a \circ b) \circ c\Big)(\lambda) = (a_1 b_1 c_1 - a_1 c_1 b_1)\ \lambda^4 + O(\lambda^5) \neq 0.$$

- However the Faà di Bruno Hopf algebra $H_{\mathrm{FdB}} = R[\mathrm{Diff}]$ lifts up to a non commutative Hopf algebra $H^{\mathrm{nc}}_{\mathrm{FdB}} = \mathbb{K}\langle x_1, x_2, ...\rangle$ with

$$\Delta^{\mathrm{nc}}_{\mathrm{FdB}}(x_n) = \sum_{m=0}^{n} x_m \otimes \sum_{(k)} x_{k_0} \cdots x_{k_m} \qquad (x_0 = 1),$$

where $(k) = (k_0, k_1, ..., k_m)$ with $k_i \geqslant 0$ and $k_0 + k_1 + \cdots + k_m = n - m$ [Brouder-AF-Krattenthaler 2006].

- The coproduct $\Delta^{\mathrm{nc}}_{\mathrm{FdB}}$ can be modified into an algebra morphism

$$\Delta^{\circledast}_{\mathrm{FdB}} : H^{\mathrm{nc}}_{\mathrm{FdB}} \longrightarrow H^{\mathrm{nc}}_{\mathrm{FdB}} \circledast H^{\mathrm{nc}}_{\mathrm{FdB}},$$

then **it represents** $\mathrm{Diff}(A)$ and of course **it loses coassociativity!**

Smooth loops

- A **loop** is a set Q with a multiplication and a unit e, such that the operators of left and right translation

$$L_a(x) = a \cdot x \qquad \text{and} \qquad R_a(x) = x \cdot a$$

are invertible, but $L_a^{-1} \neq L_{a^{-1}}$, $R_a^{-1} \neq R_{a^{-1}}$ because a^{-1} does not exist!
Call **left** and **right division**: $a \backslash b = L_a^{-1}(b)$ and $b/a = R_a^{-1}(b)$.

Smooth loops

- A **loop** is a set Q with a multiplication and a unit e, such that the operators of left and right translation

$$L_a(x) = a \cdot x \qquad \text{and} \qquad R_a(x) = x \cdot a$$

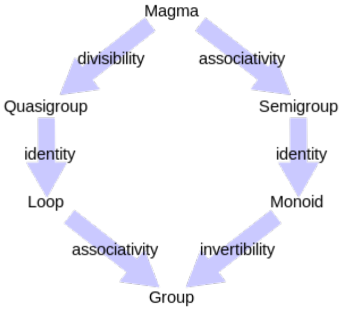
are invertible, but $L_a^{-1} \neq L_{a^{-1}}$, $R_a^{-1} \neq R_{a^{-1}}$ because a^{-1} does not exist!
Call **left** and **right division**: $a \backslash b = L_a^{-1}(b)$ and $b/a = R_a^{-1}(b)$.

- Smooth loops were introduced by Ruth Moufang [1935], later related to Maltsev algebras [1955] and to algebraic webs [Blaschke 1955].

- Any Lie group is a smooth loop:

$$a/b = a \cdot b^{-1} \quad \text{and} \quad a \backslash b = a^{-1} \cdot b.$$

- The smallest loop which is not a group is the sphere $\mathbb{S}^7$, which can be seen as the set of unit octonions in $\mathbb{O}$.



Loops, homogeneous spaces and flat connections

- **A homogeneous space is a (local) loop with the residual structure of the group action.** That is, if $M = G/H$ is a homogeneous space for a Lie group G, $p : G \to M$ is the projection and $i : U \subset M \longrightarrow G$ a (local) section around any point $e \in M$, then

$$x \cdot y = p(i(x)i(y)), \qquad x, y \in M$$

is a (local) loop multiplication [Sabinin 1972].

Loops, homogeneous spaces and flat connections

- **A homogeneous space is a (local) loop with the residual structure of the group action.** That is, if $M = G/H$ is a homogeneous space for a Lie group G, $p : G \to M$ is the projection and $i : U \subset M \longrightarrow G$ a (local) section around any point $e \in M$, then

$$x \cdot y = p(i(x)i(y)), \qquad x, y \in M$$

  is a (local) loop multiplication [Sabinin 1972].

- **A manifold with flat connection is a "geodesic" (local) loop.**
  - If Q is a smooth loop, define a parallel transport $P_a^b : T_a Q \to T_b Q$ as the differential of the map $x \mapsto b \cdot (a \backslash x)$. The tangent bundle is then trivialized, and get a flat connection ∇ [Sabinin 1986].

    N.B. For Lie groups, same result by Élie Cartan [1904, 1927], moreover torsion has zero covariant derivative!

Loops, homogeneous spaces and flat connections

- **A homogeneous space is a (local) loop with the residual structure of the group action.** That is, if $M = G/H$ is a homogeneous space for a Lie group G, $p : G \to M$ is the projection and $i : U \subset M \longrightarrow G$ a (local) section around any point $e \in M$, then

$$x \cdot y = p(i(x)i(y)), \qquad x, y \in M$$

  is a (local) loop multiplication [Sabinin 1972].

- **A manifold with flat connection is a "geodesic" (local) loop.**
  - If Q is a smooth loop, define a parallel transport $P_a^b : T_aQ \to T_bQ$ as the differential of the map $x \mapsto b \cdot (a \backslash x)$. The tangent bundle is then trivialized, and get a flat connection ∇ [Sabinin 1986].

    N.B. For Lie groups, same result by Élie Cartan [1904, 1927], moreover torsion has zero covariant derivative!

  - If M is a smoot manifold with a flat connection ∇, around any $e \in M$ can define a (local) loop by [Sabinin 1977, 1981]

$$a \bullet_e b = \exp_a \left(P_e^a (log_e(b)) \right).$$

    Moreover it is **right-alternative**: $(a \bullet b^p) \bullet b^q = a \bullet b^{p+q}$.

Loops, homogeneous spaces and flat connections

- **A homogeneous space is a (local) loop with the residual structure of the group action.** That is, if $M = G/H$ is a homogeneous space for a Lie group G, $p : G \to M$ is the projection and $i : U \subset M \longrightarrow G$ a (local) section around any point $e \in M$, then

$$x \cdot y = p(i(x)i(y)), \qquad x, y \in M$$

  is a (local) loop multiplication [Sabinin 1972].

- **A manifold with flat connection is a "geodesic" (local) loop.**
  - If Q is a smooth loop, define a parallel transport $P_a^b : T_aQ \to T_bQ$ as the differential of the map $x \mapsto b \cdot (a \backslash x)$. The tangent bundle is then trivialized, and get a flat connection ∇ [Sabinin 1986].

    N.B. For Lie groups, same result by Élie Cartan [1904, 1927], moreover torsion has zero covariant derivative!

  - If M is a smoot manifold with a flat connection ∇, around any $e \in M$ can define a (local) loop by [Sabinin 1977, 1981]

$$a \bullet_e b = \exp_a \left(P_e^a(log_e(b)) \right).$$

    Moreover it is **right-alternative**: $(a \bullet b^p) \bullet b^q = a \bullet b^{p+q}$.

  If Q is right-alternative then $\cdot = \bullet$, otherwise $a \cdot b = a \bullet \Phi(a, b)$.

Infinitesimal structure of loops: Sabinin algebras

- A **Sabinin algebra** (ex Φ-hyperalgebra) is a vector space $\mathfrak{q}$ with

$$\langle\ ;\ ,\ \rangle : T\mathfrak{q} \otimes \mathfrak{q} \wedge \mathfrak{q} \longrightarrow \mathfrak{q}$$
$$\Phi : S\mathfrak{q} \otimes S\mathfrak{q} \longrightarrow \mathfrak{q}$$

such that, if $u, v \in T\mathfrak{q}$ and $x, y, z, z' \in \mathfrak{q}$ are chosen in a given basis,

$$\langle u[z, z']v; x, y\rangle + \sum \langle u_{(1)} \langle u_{(2)}; z, z'\rangle v; y, x\rangle = 0$$
$$\sum_{(x,y,z)} \Big(\langle uz; x, y\rangle + \sum \langle u_{(1)}; \langle u_{(2)}; x, y\rangle, z\rangle \Big) = 0$$

where $\Delta u = \sum u_{(1)} \otimes u_{(2)}$ is the unshuffle coproduct on $T\mathfrak{q}$ (cocom).

Infinitesimal structure of loops: Sabinin algebras

- A **Sabinin algebra** (ex Φ-hyperalgebra) is a vector space $\mathfrak{q}$ with

$$\langle\ ;\ ,\ \rangle : T\mathfrak{q} \otimes \mathfrak{q} \wedge \mathfrak{q} \longrightarrow \mathfrak{q}$$
$$\Phi : S\mathfrak{q} \otimes S\mathfrak{q} \longrightarrow \mathfrak{q}$$

such that, if $u, v \in T\mathfrak{q}$ and $x, y, z, z' \in \mathfrak{q}$ are chosen in a given basis,

$$\langle u[z, z']v; x, y\rangle + \sum \langle u_{(1)} \langle u_{(2)}; z, z'\rangle v; y, x\rangle = 0$$
$$\sum_{(x,y,z)} \Big(\langle uz; x, y\rangle + \sum \langle u_{(1)}; \langle u_{(2)}; x, y\rangle, z\rangle \Big) = 0$$

where $\Delta u = \sum u_{(1)} \otimes u_{(2)}$ is the unshuffle coproduct on $T\mathfrak{q}$ (cocom).

- **Geometrical explanation**: if $\mathfrak{q} = T_e Q$ and ∇ is the flat connection on Q, can choose a basis of ∇-constant vector fields $X, Y, Z, \ldots$ so that $\nabla_X Y = 0$ and $R(X, Y)Z = 0$, and set

$$\langle\ Z_1, \ldots, Z_m; X, Y\rangle = \nabla_{Z_1} \cdots \nabla_{Z_m} T(X, Y)$$

(Φ omitted because more complicated). Then
Sabinin identities = Bianchi identities relating torsion and curvature.

Smooth and (pro)algebraic loops on commutative algebras

Loop

Q smooth
or (pro)algebraic

$Q(A) \cong \operatorname{Hom}_{uCom}(R[Q], A)$
convolution group

reps ? →

← algebraic loop

Function algebra

$R[Q] = \mathcal{O}(Q)$

alg: as + com
coalg: mag + codivisions

infinitesimal structure ↓

Hopf-type duality ↕

Sabinin algebra

$\mathfrak{q} = T_e Q \cong \mathfrak{X}_L(Q)$
$\subset \operatorname{Prim} U\mathfrak{q}$

$uMag \to Sab : A \mapsto A_S$
[Shestakov-Umirbaev 2002]

algebra ext. →

← primitives

↔ adjoint functors

Enveloping algebra

$U\mathfrak{g} \cong R[G]^*$
alg: mag + divisons
coalg: cocom + coas

$\operatorname{Hom}_{Sab}(\mathfrak{q}, A_S) \cong \operatorname{Hom}_{uMag}(U\mathfrak{q}, A)$

Loop of formal diffeomorphisms

Standard way to produce loops: invertibles in magmatic algebras or formal loops.
Here, non standard one: modify coefficients [AF-Shestakov]

- **Heisenberg loop**: the set of Heisenberg matrices (or any triangular)

$$HL_3(A) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in A \right\}$$

  is a loop with matrix product even when A is a **non-associative algebra** (e.g. octonions). It is a group if A associative (e.g. quaternions).

Loop of formal diffeomorphisms

Standard way to produce loops: invertibles in magmatic algebras or formal loops.
Here, non standard one: modify coefficients [AF-Shestakov]

- **Heisenberg loop**: the set of Heisenberg matrices (or any triangular)

$$HL_3(A) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in A \right\}$$

  is a loop with matrix product even when A is a **non-associative algebra** (e.g. octonions). It is a group if A associative (e.g. quaternions).

- **Loop of formal diffeomorphisms**: the set of formal diffeomorphism

$$\mathrm{Diff}(A) = \Big\{ a = \sum_{n\geqslant 0} a_n \, \lambda^{n+1} \mid a_0 = 1, a_n \in A \Big\},$$

  with composition

$$a \circ b = \sum_{n\geqslant 0} \sum_{m=0}^{n} \sum_{k_0+\cdots+k_m=n-m} a_m \, b_{k_0} \cdots b_{k_m} \, \lambda^{n+1}$$

  is a loop if A is a unital **associative** algebra. It is **right alternative** and therefore **power associative**. It is a group if A is commutative.

Loop of formal diffeomorphisms

Standard way to produce loops: invertibles in magmatic algebras or formal loops.
Here, non standard one: modify coefficients [AF-Shestakov]

- **Heisenberg loop**: the set of Heisenberg matrices (or any triangular)

$$HL_3(A) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in A \right\}$$

  is a loop with matrix product even when A is a **non-associative algebra** (e.g. octonions). It is a group if A associative (e.g. quaternions).

- **Loop of formal diffeomorphisms**: the set of formal diffeomorphism

$$\mathrm{Diff}(A) = \Big\{ a = \sum_{n \geqslant 0} a_n \, \lambda^{n+1} \mid a_0 = 1, a_n \in A \Big\},$$

  with composition

$$a \circ b = \sum_{n \geqslant 0} \sum_{m=0}^{n} \sum_{k_0 + \cdots + k_m = n-m} a_m \; b_{k_0} \cdots b_{k_m} \; \lambda^{n+1}$$

  is a loop if A is a unital **associative** algebra. It is **right alternative** and therefore **power associative**. It is a group if A is commutative.

- **Loop of $\mathcal{P}$-expanded series**: the same holds for series expanded over any operad $\mathcal{P}$ with $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = \{\mathrm{id}\}$ and coeff in A.

Proof that the free product $\circledast$ is necessary

In the loop $\mathrm{Diff}(A)$, call b^{-1} the series as if A were commutative, then

$$a/b = a \circ b^{-1} \quad \text{but} \quad b\backslash a \neq b^{-1} \circ a \ !$$

Proof that the free product $\circledast$ is necessary

In the loop $\mathrm{Diff}(A)$, call b^{-1} the series as if A were commutative, then

$$a/b = a \circ b^{-1} \quad \text{but} \quad b\backslash a \neq b^{-1} \circ a \ !$$

- In the series $b\backslash a$, the coefficient

$$(b\backslash a)_3 = a_3 - \left(2b_1 a_2 + b_1 a_1^2\right) + \left(5b_1^2 a_1 + b_1 a_1 b_1 - 3b_2 a_1\right) - \left(5b_1^3 - 2b_1 b_2 - 3b_2 b_1 + b_3\right)$$

  contains the term $b_1 a_1 b_1$ which can not be represented in the form $f(b) \otimes g(a) \in H^{\mathrm{nc}}_{\mathrm{FdB}} \otimes H^{\mathrm{nc}}_{\mathrm{FdB}}$, while clearly belongs to $H^{\mathrm{nc}}_{\mathrm{FdB}} \circledast H^{\mathrm{nc}}_{\mathrm{FdB}}$.

  This **justifies the need to replace $\otimes$ by $\circledast$** in the definition of the coproduct of $R[\mathrm{Diff}(A)]$.

Proof that the free product $\circledast$ is necessary

In the loop $\mathrm{Diff}(A)$, call b^{-1} the series as if A were commutative, then

$$a/b = a \circ b^{-1} \quad \text{but} \quad b\backslash a \neq b^{-1} \circ a \,!$$

- In the series $b\backslash a$, the coefficient

$$(b\backslash a)_3 = a_3 - \big(2b_1a_2 + b_1a_1^2\big) + \big(5b_1^2a_1 + b_1a_1b_1 - 3b_2a_1\big) - \big(5b_1^3 - 2b_1b_2 - 3b_2b_1 + b_3\big)$$

  contains the term $b_1a_1b_1$ which can not be represented in the form $f(b) \otimes g(a) \in H^{\mathrm{nc}}_{\mathrm{FdB}} \otimes H^{\mathrm{nc}}_{\mathrm{FdB}}$, while clearly belongs to $H^{\mathrm{nc}}_{\mathrm{FdB}} \circledast H^{\mathrm{nc}}_{\mathrm{FdB}}$.

  This **justifies the need to replace $\otimes$ by $\circledast$** in the definition of the coproduct of $R[\mathrm{Diff}(A)]$.

- Moreover, the difference

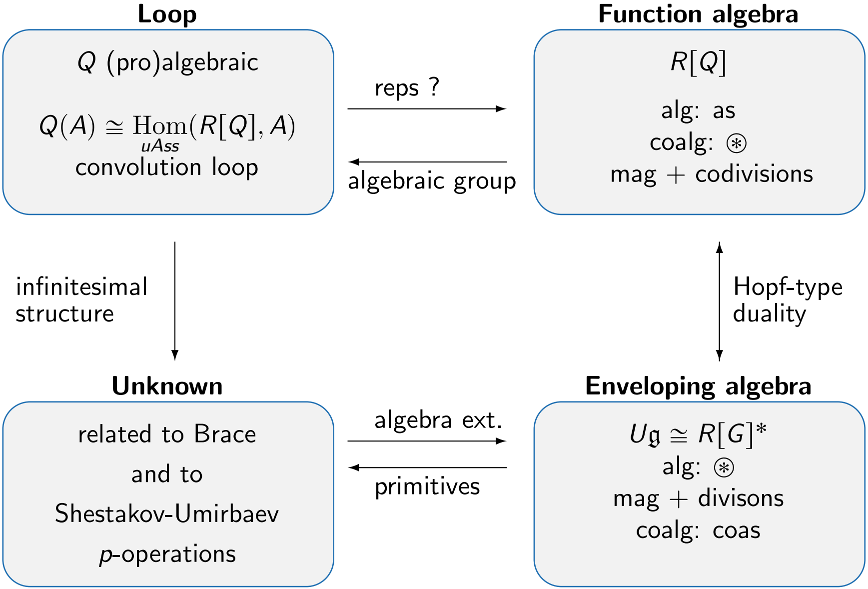
$$\big(a/b - b\backslash a\big)_3 = b_1^2a_1 - b_1a_1b_1$$

  shows **why the non-comm. Faà di Bruno Hopf algebra exists**: $\Delta^{\mathrm{nc}}_{\mathrm{FdB}}$ recovered from $\Delta^{\circledast}_{\mathrm{FdB}}$ by composing with the projection

$$H^{\mathrm{nc}}_{\mathrm{FdB}} \circledast H^{\mathrm{nc}}_{\mathrm{FdB}} \to H^{\mathrm{nc}}_{\mathrm{FdB}} \otimes H^{\mathrm{nc}}_{\mathrm{FdB}}$$

  which identifies $b_1a_1b_1$ and $b_1^2a_1$. Then $a/b = b\backslash a$ and b^{-1} is a two-sided inverse.

(Pro)algebraic loops on non-commutative algebras [AF-IS]



THANK YOU!