φ-deformed shuffle bialgebras and renormalization

V.C. Bùi, G.H.E. Duchamp, Hoang Ngoc Minh, Q.H. Ngô

Paths to, from and in renormalization
February, 8th-12th 2016, Potsdam

Plan

1. Introduction
   1.1 Renormalization of (all) divergent polyzetas ?
   1.2 Structures of harmonic sums and of polylogarithms at positive indices
   1.3 Shifts and derivations of shuffle-conc bialgebra
2. Background on combinatorics of φ-deformed of shuffle bialgebras
   2.1 $(A\langle Y\rangle, ., 1_{Y^*}, \Delta_{\text{ш}}, \epsilon_Y)$ and its deformations
   2.2 φ-shuffle products
   2.3 φ-extended Schützenberger's factorization in quasi-shuffle-conc bialgebras
3. Abel like theorems for noncommutative generating series
   3.1 Global renormalization of polyzetas at positive indices
   3.2 Global renormalization of polyzetas at negative indices
   3.3 Bi-integro-differential algebra $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$
4. Some consequences on structure of polyzetas
   4.1 Homogenous polynomials relations among local coordinates
   4.2 Noetherian rewriting system and irreducible local coordinates
   4.3 Examples irreducible local coordinates by computer

INTRODUCTION

Renormalization of (all) divergent polyzetas ?

For $z \in \mathbb{C}$, $|z| < 1$, let $(s_1, \ldots, s_r) \in \mathbb{C}^r$, $r \in \mathbb{N}_+$, the polylogarithm is well defined

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) := \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}}.$$

Then the Taylor expansion of $(1-z)^{-1} \mathrm{Li}_{s_1,\ldots,s_r}(z)$ is given by

$$\frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}_{s_1,\ldots,s_r}(N)\, z^N,$$

where the coefficient $\mathrm{H}_{s_1,\ldots,s_r}(N)$ is a harmonic sum which can be expressed as follows

$$\mathrm{H}_{s_1,\ldots,s_r}(N) := \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}.$$

For any $m = 1, .., r$, if $\sum_{i=1}^{m} \Re(s_i) > 1$ then, after a theorem by Abel, one obtains[1] the polyzeta as

$$\lim_{z \to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z) = \lim_{N \to \infty} \mathrm{H}_{s_1,\ldots,s_r}(N) = \zeta(s_1, \ldots, s_r) := \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}$$

else ???

[1] see the talk of Guo.

Encoding multi-indices by words

Let X^* and Y^* be the free **monoids** (admitting 1_{X^*} and 1_{Y^*} as units) generated respectively by $X = \{x_0, x_1\}$ and $Y = \{y_k\}_{k \geq 1}$. Here, we suppose that[2] $(s_1, \ldots, s_r) \in \mathbb{N}_+^r$.

$$\mathbf{s} = (s_1, \ldots, s_r) \leftrightarrow u = y_{s_1} \ldots y_{s_r} \rightleftharpoons_{\pi_Y}^{\pi_X} v = x_0^{s_1 - 1} x_1 \ldots x_0^{s_r - 1} x_1.$$

For $s_1 > 1$, the associated **words** in $x_0 X^* x_1$ or $(Y - \{y_1\}) Y^*$ are said to be convergent. For $r \geq k \geq 1$, a divergent word is of the following form

$$(\{1\}^k, s_{k+1}, \ldots, s_r) \leftrightarrow y_1^k y_{s_{k+1}} \ldots y_{s_r} \rightleftharpoons_{\pi_Y}^{\pi_X} x_1^k x_0^{s_{k+1} - 1} x_1 \ldots x_0^{s_r - 1} x_1.$$

Let Y_0^* be the free **monoid** generated by $Y_0 = Y \cup \{y_0\}$ with $1_{Y_0^*}$ as unit.

$$(s_1, \ldots, s_r) \in \mathbb{N}^r \leftrightarrow y_{s_1} \ldots y_{s_r} \in Y_0^*.$$

The **length** and the **weight** of $w = y_{s_1} \ldots y_{s_r} \in Y^*$ or Y_0^* (resp. $w = x_{s_1} \ldots x_{s_r} \in X^*$) are respectively $|w| = r$, for Y^* or Y_0^*, (resp. X^*) and $(w) = s_1 + \ldots + s_r$, for Y^* and Y_0^*.

Let $\mathcal{L}ynY_0, \mathcal{L}ynY$ and $\mathcal{L}ynX$ denote the sets of Lyndon words respectively over Y_0, Y and X, totally ordered by $x_0 < x_1$ and $y_0 > y_1 > y_2, \cdots$.

[2] see the minicourses of Ebrahimi-Fard and Singer.

Indexing polylogarithms and harmonic sums by words

Let $(s_1, \ldots, s_r) \in \mathbb{N}_+^r$. Then

$$\begin{array}{rcl} \mathrm{Li}_{s_1,\ldots,s_r}(z) & = & \mathrm{Li}_{y_{s_1}\ldots y_{s_r}}(z) & = & \mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1}(z), \\ \mathrm{H}_{s_1,\ldots,s_r}(N) & = & \mathrm{H}_{y_{s_1}\ldots y_{s_r}}(N) & = & \mathrm{H}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1}(N), \\ \zeta(s_1,\ldots,s_r) & = & \zeta(y_{s_1}\ldots y_{s_r}) & = & \zeta(x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1). \end{array}$$

Let $\mathcal{Z}$ denote the $\mathbb{Q}$-algebra generated by convergent polyzetas.
Let $(s_1, \ldots, s_r) \in \mathbb{N}^r$. Then[3]

$$\begin{array}{rcl} \mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}}(z) & := & \mathrm{Li}_{-s_1,\ldots,-s_r}(z) & = & \sum_{n_1>\ldots>n_r>0} n_1^{s_1}\ldots n_r^{s_r}\, z^{n_1}, \\ \mathrm{H}^-_{y_{s_1}\ldots y_{s_r}} & := & \mathrm{H}_{-s_1,\ldots,-s_r} & = & \sum_{N\geq n_1>\ldots>n_r>0} n_1^{s_1}\ldots n_r^{s_r}, \\ \zeta^-(y_{s_1}\ldots y_{s_r}) & := & \zeta(-s_1,\ldots,-s_r) & \leftrightarrow & \sum_{n_1>\ldots>n_r>0} n_1^{s_1}\ldots n_r^{s_r}. \end{array}$$

[3] Previous works on renormalization of $\zeta(-s_1, \ldots, -s_r)$:

- **D. Manchon, S. Paycha**, Nested sums of symbols and renormalised multiple zeta functions, 2010.
- **L. Guo, B. Zhang**, *Differential Birkhoff decomposition and the renormalization of multiple zeta values*, 2012.
- **H. Furusho, Y. Komori, K. Matsumoto, H. Tsumura**, *Desingularization of multiple zeta-functions of generalized Hurwitz-Lerch type*, 2014.

Harmonic sums as monomial quasi-symmetric functions

Let $N, r \in \mathbb{N}, r > 0$ and let $\mathbf{s} = (s_1, \ldots, s_r) \in (\mathbb{N}_+)^*$ be the multi-index associated to the word $w = y_{s_1} \ldots y_{s_r} \in Y^*$.
Using the correspondence

$$(\mathbb{N}_+)^* \ni (s_1, \ldots, s_r) = \mathbf{s} \quad \leftrightarrow \quad w = y_{s_1} \ldots y_{s_r} \in Y^*,$$

the monomial quasi-symmetric functions, on $\mathbf{t} = \{t_i\}_{i \geq 1}$, are defined by

$$M_{1_{Y^*}}(\mathbf{t}) = M_{\emptyset}(\mathbf{t}) = 1 \quad \text{and} \quad M_w(\mathbf{t}) = M_{\mathbf{s}}(\mathbf{t}) = \sum_{n_1 > \ldots > n_k > 0} t_{n_1}^{s_1} \ldots t_{n_k}^{s_k}.$$

For any $u, v \in X^*$, one has (**Knutson**'s inner product, 1973)

$$M_{u \,\text{⧢}\, v}(\mathbf{t}) = M_u(\mathbf{t}) M_v(\mathbf{t}).$$

$\mathrm{H}_{s_1, \ldots, s_r}(N)$ (resp. $\mathrm{H}_{-s_1, \ldots, -s_r}(N)$) is obtained then by specializing the indeterminates $\mathbf{t} = \{t_i\}_{i \geq 1}$ in the monomial quasi-symmetric function $M_{\mathbf{s}}(\mathbf{t}) = M_w(\mathbf{t})$ respectively as follows (**Hoffman**, 1997)

$$t_i = 1/i \quad (\text{resp. } t_i = i) \quad \text{and } \forall i > N, t_i = 0.$$

Hence, $\mathbb{Q}\{\mathrm{H}_w\}_{w \in Y^*} \cong \mathbb{Q}\langle Y \rangle$ and (**HNM**, 2003)

$$(\mathbb{Q}\{\mathrm{H}_w\}_{w \in Y^*}, \times) \cong (\mathbb{Q}\langle Y \rangle, \text{⧢}) \cong (\mathbb{Q}[\mathcal{L}ynY], \text{⧢}).$$

Polylogarithms as iterated path integrals

The iterated integral, associated to $w \in X^*$, along the path $z_0 \rightsquigarrow z$ and over the differential forms $\omega_0(z) = dz/z$ and $\omega_1(z) = dz/(1-z)$, is defined, on any appropriate simply connected domain Ω, as follows[4].

$$\alpha_{z_0}^z(w) = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*}, \\ \displaystyle\int_{z_0}^z \omega_{i_1}(t)\alpha_{z_0}^t(u) & \text{if} \quad w = x_{i_1}u, \quad x_{i_1} \in X, u \in X^*. \end{cases}$$

For any $u, v \in X^*$, one has (**Chen**'s lemma, 1954)

$$\alpha_{z_0}^z(u \sqcup\!\sqcup v) = \alpha_{z_0}^z(u)\alpha_{z_0}^z(v).$$

$\{\mathrm{Li}_w\}_{w \in X^* x_1}$ are obtained then as iterated integrals

$$\forall w \in X^* x_1, \qquad \mathrm{Li}_w(z) = \alpha_0^z(w).$$

Setting $\mathrm{Li}_{x_0}(z) = \log z = \alpha_1^z(x_0)$, one can use iterated integrals, with $z_0 = 0$, to calculate the other values, via a theorem by **Radford** (1956) because $\mathbb{Q}\{\mathrm{Li}_w\}_{w \in X^*} \cong \mathbb{Q}\langle X \rangle$ and (**HNM, Petitot, Hoeven**, 1998)

$$(\mathbb{Q}\{\mathrm{Li}_w\}_{w \in X^*}, \times) \cong (\mathbb{Q}\langle X \rangle, \sqcup\!\sqcup) \cong (\mathbb{Q}[\mathcal{L}ynX], \sqcup\!\sqcup)$$

(or more generally **Deneufchâtel, Duchamp, HNM, Solomon**, 2011).

[4] see the talk of Panzer.

Derivations and shifts in shuffle algebras

Definition

Let $S \in \mathbb{C}\langle\langle X \rangle\rangle$ (resp. $\mathbb{C}\langle X \rangle$) and $P \in \mathbb{Q}\langle X \rangle$ (resp. $\mathbb{C}\langle\langle X \rangle\rangle$). The *right* (resp. *left*) *residual* of P by S, is $P \triangleleft S$ (resp. $S \triangleright P$) defined by[5] :

$$\forall w \in X^*, \quad \langle P \triangleleft S | w \rangle = \langle P | S w \rangle \quad (\text{resp. } \langle S \triangleright P | w \rangle = \langle P | w S \rangle).$$

In particular, for any $x, y \in X$ and $w \in X^*$, $x \triangleright (wy) = (yw) \triangleleft x = \delta_x^y w$.

These residuals (or shifts) are associative and commute with each other :

$$S \triangleright (P \triangleleft R) = (S \triangleright P) \triangleleft R, \quad P \triangleleft (RS) = (P \triangleleft R) \triangleleft S, \quad (RS) \triangleright P = R \triangleright (S \triangleright P).$$

Proposition (derivations and automorphisms)

Let $P \in \mathbb{C}\langle X \rangle$ (resp. $\mathbb{C}\langle\langle X \rangle\rangle$) and $T \in \mathbb{C}\langle\langle X \rangle\rangle$ (resp. $\mathbb{C}\langle X \rangle$) such that $\Delta_{ш}(T) = 1 \otimes T + T \otimes 1$. Then

- *$P \mapsto P \triangleleft T$ and $P \mapsto T \triangleright P$ are derivations of $(\mathbb{C}\langle X \rangle, ш, 1_{X^*})$ (resp. $(\mathbb{C}\langle\langle X \rangle\rangle, ш, 1_{X^*})$).*
- *$P \mapsto P \triangleleft \exp(tT)$ and $P \mapsto \exp(tT) \triangleright P$ are one-parameter groups of automorphisms of $(\mathbb{C}\langle X \rangle, ш, 1_{X^*})$ (resp. $(\mathbb{C}\langle\langle X \rangle\rangle, ш, 1_{X^*})$).*

---

[5] These actions are the shifts of functions in harmonic analysis.

COMBINATORICS OF φ-SHUFFLE-CONC BIALGEBRAS

$(A\langle Y\rangle, ., 1_{Y^*}, \Delta_{ш}, \epsilon_Y)$ and its deformations

A : commutative and associative algebra with unit over $\mathbb{Q}$.
Let $A\langle Y\rangle$ and $A\langle\langle Y\rangle\rangle$ denote the sets of polynomials and of formal power series over Y, with coefficients in A, equipped with the concatenation.
They are also endowed with the φ-shuffle defined recursively by

$$\begin{cases} \forall w \in Y^*, & w \uplus_\varphi 1_{Y^*} & = & 1_{Y^*} \uplus_\varphi w = w, \\ \forall y_i, y_j \in Y, \forall u, v \in Y^*, & y_i u \uplus_\varphi y_j v & = & y_i(u \uplus_\varphi y_j v) + y_j(y_i u \uplus_\varphi v) \\ & & + & \varphi(y_i, y_j)(u \uplus_\varphi v), \end{cases}$$

where φ is an arbitrary mapping defined by its structure constants

$$\varphi : Y \times Y \longrightarrow AY, \qquad (y_i, y_j) \longmapsto \sum_{k \in I \subset \mathbb{N}_+} \gamma_{i,j}^k \, y_k.$$

It is said to be dualizable if there exists $\Delta_{\uplus_\varphi} : A\langle Y\rangle \to A\langle Y\rangle \otimes A\langle Y\rangle$ such that the dual mapping $\Big(A\langle Y\rangle \otimes A\langle Y\rangle\Big)^* \to A\langle\langle Y\rangle\rangle$ restricts to $\uplus_\varphi$.

Theorem (Duchamp, Enjalbert, HNM, Tollu, 2014)

1. *The law $\uplus_\varphi$ is associative (resp. commutative) if and only if the linear extension $\varphi : AY \otimes AY \longrightarrow AY$ is so.*
2. *Let $\gamma_{x,y}^z := \langle \varphi(x,y) | z\rangle$ be the structure constants of φ, then $\uplus_\varphi$ is dualizable if and only if $(\gamma_{x,y}^z)_{x,y,z \in Y}$ has the following property*

$$(\forall z \in Y)(\#\{(x,y) \in Y^2 | \gamma_{x,y}^z \neq 0\} < +\infty).$$

Examples

Name	(recursion) Formula	φ
Shuffle	$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v)$	$\varphi \equiv 0$
Stuffle	$x_i u \uplus x_j v = x_i(u \uplus x_j v) + x_j(x_u \uplus v) + x_{i+j}(u \uplus v)$	$\varphi(x_i, x_j) = x_{i+j}$
Min-shuffle	$x_i u \uplus x_j v = x_i(u \uplus x_j v) + x_j(x_u \sqcup\!\sqcup v) - x_{i+j}(u \uplus v)$	$\varphi(x_i, x_j) = -x_{i+j}$
Muffle	$x_i u \sqcup\!\sqcup x_j v = x_i(u \sqcup\!\sqcup x_j v) + x_j(x_u \sqcup\!\sqcup v) + x_{i \times j}(u \sqcup\!\sqcup v)$	$\phi(x_i, x_j) = x_{i \times j}$
q-stuffle	$x_i u \uplus_q x_j v = x_i(u \uplus_q x_j v) + x_j(x_u \uplus_q v) + q x_{i+j}(u \sqcup\!\sqcup v)$	$\varphi(x_i, x_j) = q x_{i+j}$
q-shuffle	$x_i u \sqcup\!\sqcup_q x_j v = x_i(u \sqcup\!\sqcup_q x_j v) + x_j(x_u \sqcup\!\sqcup_q v) + q^{i \times j} x_{i+j}(u \sqcup\!\sqcup v)$	$\varphi(x_i, x_j) = q^{i \times j} x_{i+j}$
$\mathtt{LDIAG}(1, q_s)$ non-crossed, non-shifted	$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v) + q_s^{\lvert a \rvert \lvert b \rvert}(a.b)(u \sqcup\!\sqcup v)$	$\varphi(a, b) = q_s^{\lvert a \rvert \lvert b \rvert}(a.b)$
B-shuffle	$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v) + \langle a, b \rangle (u \sqcup\!\sqcup v)$	$\varphi(a, b) = \langle a, b \rangle = \langle b, a \rangle$
Semigroup-shuffle	$x_t u \sqcup\!\sqcup_\perp x_s v = x_t(u \sqcup\!\sqcup_\perp x_s v) + x_s(x_t u \sqcup\!\sqcup_\perp v) + x_{t \perp s}(u \sqcup\!\sqcup_\perp v)$	$\varphi(x_t, x_s) = x_{t \perp s}$

Properties of φ-deformed shuffle products

Theorem (Bùi, Duchamp, HNM, Ngô, Tollu, 2014)

φ is supposed dualizable. We still denote the dual law of $⧢_\varphi$ by

$$\Delta_{⧢_\varphi} : A\langle Y\rangle \longrightarrow A\langle Y\rangle \otimes A\langle Y\rangle.$$

$\mathcal{B}_\varphi := (A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{⧢_\varphi}, \varepsilon)$ *is then a bialgebra.*
Moreover, if φ is commutative, the following conditions are equivalent

1. $\mathcal{B}_\varphi$ *is an enveloping bialgebra.* (**CQMM** *theorem*)
2. $\mathcal{B}_\varphi$ *is isomorphic to* $(A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{⧢}, \epsilon)$ *as a bialgebra.*
3. *For all $y \in Y$, the following series is a polynomial.*

$$\pi_1^\varphi(y) = y + \sum_{l\geq 2} \frac{(-1)^{l-1}}{l} \sum_{x_1,\ldots,x_l\in Y} \langle y|\varphi(x_1,\ldots,x_l)\rangle\, x_1 \ldots x_l.$$

In the previous equivalent cases, φ is called moderate.
From now on, we suppose φ ass., com., dualizable and moderate.

Schützenberger factorization in $(A\langle Y\rangle, ., 1_{Y^*}, \Delta_{\uplus_\varphi}, \epsilon_Y)$

Proposition (isomorphism of bialgebras (φ as above))

Let $\Phi : A\langle Y\rangle \longrightarrow A\langle Y\rangle$ be the conc*-morphism defined by*[6] $\pi_1^{\varphi}(y_k)$:

$$\forall y \in Y, \quad \Phi(y) = \pi_1^{\varphi}(y) = y + \sum_{l\geq 2} \frac{(-1)^{l-1}}{l} \sum_{x_1,\ldots,x_l\in Y} \gamma^{y}_{x_1,\ldots,x_l}\, x_1\ldots x_l,$$

$$\gamma^{y}_{x_1,\ldots,x_l} = \sum_{t_1,\ldots,t_{l-2}\in Y} \gamma^{y}_{x_1,t_1}\gamma^{t_1}_{x_2,t_2}\ldots\gamma^{t_{l-2}}_{x_{l-1},x_l}.$$

Then Φ is a bialgebra isomorphism from $(A\langle Y\rangle, \mathrm{conc}, \Delta_{\uplus_\varphi}, \epsilon_Y)$ to $(A\langle Y\rangle, \mathrm{conc}, \Delta_{\sqcup\!\sqcup}, \epsilon_Y)$.

Definition (PBW-Lyndon basis and its dual basis)

$$\Pi_{y_k} = \pi_1^{\varphi}(y_k), \quad \text{for } k \geq 1,$$
$$\Pi_l = [\Pi_s, \Pi_r], \quad \text{for } l \in \mathcal{L}ynY, \text{ standard factorization of } l = (s, r),$$
$$\Pi_w = \Pi_{l_1}^{i_1}\ldots\Pi_{l_k}^{i_k}, \quad \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, l_1 > \ldots > l_k, l_1\ldots, l_k \in \mathcal{L}ynY.$$

$\{\Sigma_w\}_{w\in Y^*}$ = dual basis of $\{\Pi_w\}_{w\in Y^*}$: $\forall u, v \in Y^*, \langle\Sigma_v|\Pi_u\rangle = \delta_{u,v}$.

For any $w = l_1^{i_1}\ldots l_k^{i_k}$, with $l_1,\ldots,l_k \in \mathcal{L}ynY$ and $l_1 > \ldots > l_k$,

$$\Sigma_w = \frac{1}{i_1!\ldots i_k!}\Sigma_{l_1}^{\uplus_\varphi i_1} \uplus_\varphi \ldots \uplus_\varphi \Sigma_{l_k}^{\uplus_\varphi i_k} \in A\langle Y\rangle.$$

Theorem (Bùi, Duchamp, HNM, Ngô, Tollu, 2014)

$$\mathcal{D}_Y = \sum_{w\in Y^*} w\otimes w = \sum_{w\in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathcal{L}ynY} e^{\Sigma_l\otimes\Pi_l}.$$

[6] π_1^{φ} is the linear endomorphism of $A\langle\langle Y\rangle\rangle$ given by the logarithm of $\mathcal{D}_Y$.

ABEL LIKE THEOREMS
FOR NONCOMMUTATIVE GENERATING SERIES

Noncommutative generating series of polyzetas

$$\mathrm{L} := \sum_{w\in X^*=\{x_0,x_1\}^*} \mathrm{Li}_w \; w = (\mathrm{Li}_\bullet \otimes \mathrm{Id})\mathcal{D}_X = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l}\, P_l},$$

$$\mathrm{H} := \sum_{w\in Y^*=\{y_k\}_{k\geq 1}^*} \mathrm{H}_w \; w = (\mathrm{H}_\bullet \otimes \mathrm{Id})\mathcal{D}_Y = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\Sigma_l}\Pi_l}.$$

$$Z_{ш} := \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\zeta(S_l)P_l} \quad \text{and} \quad Z_{⧢} := \prod_{l\in\mathcal{L}ynY-\{y_1\}}^{\searrow} e^{\zeta(\Sigma_l)\Pi_l}.$$

$\mathrm{L}, Z_{ш}$ are group-like, for $\Delta_{ш}$, and $\mathrm{H}, \mathrm{H}^-, Z_{⧢}$ are group-like, for $\Delta_{⧢}$.

$$\begin{array}{rcll} (DE) \qquad d\mathrm{L} & = & (\omega_0\, x_0 + \omega_1\, x_1)\mathrm{L}, & \\ \mathrm{Gal}_{\mathbb{C}}(DE) & = & \{e^C \mid C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle\} & \quad (\textbf{HNM}, 2003). \end{array}$$

Let $dm(A) := \{\overline{Z}_{ш} = Z_{ш} e^C \mid C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle, \langle e^C|x_0\rangle = \langle e^C|x_1\rangle = 0\}$.
Then $dm(A) = \mathrm{Gal}_{\mathbb{C}}^{\geq 2}(DE)$ is a strict normal subgroup of $\mathrm{Gal}_{\mathbb{C}}(DE)$.

First global renormalization of divergent polyzetas

Let $e^C \in \mathrm{Gal}_{\mathbb{C}}(DE)$. Putting $\overline{\mathrm{L}} := \mathrm{L}e^C$ and $\overline{Z}_{Ш} := Z_{Ш}e^C$, one has

$$\overline{\mathrm{L}}(z)\widetilde{_{z\to 1}}\exp[-x_1\log(1-z)]\overline{Z}_{Ш}, \quad \overline{\mathrm{H}}(N)\widetilde{_{N\to\infty}}\exp\left[-\sum_{k\geq 1}\mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right]\pi_Y\overline{Z}_{Ш}.$$

Theorem (Abel like theorem, HNM, 2009)

$$\lim_{z\to 1}\exp\left[-y_1\log\frac{1}{1-z}\right]\pi_Y\overline{\mathrm{L}}(z) = \lim_{N\to\infty}\exp\left[\sum_{k\geq 1}\mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right]\overline{\mathrm{H}}(N) = \pi_Y\overline{Z}_{Ш}.$$

Let $\{\overline{\gamma}_w\}_{w\in Y^*}$ be the finite parts of $\{\overline{\mathrm{H}}_w\}_{w\in Y^*}$ and let

$$\overline{Z}_\gamma := \sum_{w\in Y^*}\overline{\gamma}_w\, w.$$

Then, for $\Delta_{ш\!\!\!+}$, $\overline{\gamma}_\bullet$ is a character and $\overline{Z}_\gamma$ is group-like (**HNM**, 2009).
In particular, remarking $\overline{Z}_{Ш} \in dm(A)$ and denoting Γ the Euler's Gamma function, the factorization and the Abel like theorem yield respectively

$$\overline{Z}_\gamma = e^{\gamma y_1}\overline{Z}_{ш\!\!\!+} \quad \text{and} \quad \overline{Z}_\gamma = \Gamma(y_1+1)\pi_Y\overline{Z}_{Ш}.$$

Hence, by **cancellation**, one gets finally

$$\overline{Z}_{ш\!\!\!+} = \mathrm{Mono}(y_1)\pi_Y\overline{Z}_{Ш}, \quad \text{where} \quad \mathrm{Mono}(y_1) = \exp\left[-\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\right].$$

Therefore, if $\gamma \notin A$ then γ is transcendental over the A-algebra generated by convergent polyzetas (**HNM**, 2009).

Euler-Mac Laurin constants associated to polyzetas

Now, for $e^C = 1_{X^*}$, extracting the coefficients in L and H, for any $w \in Y^*, k \in \mathbb{N}_+$, there exists (**Costermans, Enjalbert, HNM**, 2004)

- $a_i, b_{i,j} \in \mathcal{Z}$, such that

$$\mathrm{Li}_w(z) \underset{z\to 1}{\asymp} \sum_{i=1}^{|w|} a_i \log^i(1-z) + \langle Z_{ш}|w\rangle + \sum_{i=1}^{+\infty} b_{i,j}(1-z)^j \log^i(1-z).$$

- $\gamma_w, \alpha_i, \beta_{i,j} \in \mathcal{Z}[\gamma]$, such that

$$\mathrm{H}_w(N) \underset{N\to+\infty}{\asymp} \sum_{i=1}^{|w|} \alpha_i \log^i(N) + \gamma_w + \sum_{j=1}^{+\infty} \beta_{i,j} \frac{\log^i(N)}{N^j}.$$

Identifying the coefficients in

$$Z_\gamma = \Gamma(y_1+1)\pi_Y Z_{ш}, \quad \text{where} \quad Z_\gamma := \sum_{w\in Y^*} \gamma_w\ w,$$

we get (**Costermans, HNM**, 2005)

$$\gamma_{y_1^k} = \sum_{s_1,\ldots,s_k>0, s_1+\ldots+ks_k=k} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\left(-\frac{\zeta(2)}{2}\right)^{s_2}\ldots\left(-\frac{\zeta(k)}{k}\right)^{s_k},$$

$$\gamma_{y_1^k w} = \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} ш \pi_X w])}{i!}\left(\sum_{j=1}^{i} b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\right),$$

where $k \in \mathbb{N}_+, w \in Y^+$ and $b_{n,k}(t_1,\ldots,t_k)$ are Bell polynomials.

Li^-_w and H^-_w as polynomials

Proposition

For any $w \in Y_0^*$, $\mathrm{Li}^-_w(z) \in \mathbb{Q}[(1-z)^{-1}] \subsetneq \mathcal{C}$ *and* $\mathrm{H}^-_w(N) \in \mathbb{Q}[N]$ *of degree* $|w| + (w)$ *and of valuation* 1.
Hence, there exists $C^-_w \in \mathbb{Q}$ *and* $B^-_w \in \mathbb{N}$ *such that*

$$\mathrm{H}^-_w(N) \underset{N\to+\infty}{\widetilde{\quad}} C^-_w N^{|w|+(w)} \quad \text{and} \quad \mathrm{Li}^-_w(z) \underset{z\to 1}{\widetilde{\quad}} B^-_w/(1-z)^{|w|+(w)},$$

$$C^-_w = \prod_{w=uv, v\neq 1_{Y_0^*}} \frac{1}{(v) + |v|} \quad \text{and} \quad B^-_w = (|w| + (w))!\, C^-_w .$$

Example (of H^-_w and Li^-_w)

$$\mathrm{H}^-_{y_2y_2}(N) = \frac{1}{180} N(10N^5 + 12N^4 - 10N^3 - 35N^2 + 5N + 3),$$
$$\mathrm{H}^-_{y_2y_3}(N) = \frac{1}{8} N^2(N-1)(2N^2 + N - 2)(N+1)^2,$$
$$\mathrm{Li}^-_{y_1y_1}(z) = -(1-z)^{-1} + 3(1-z)^{-2} + 3(1-z)^{-3} - (1-z)^{-4},$$
$$\mathrm{Li}^-_{y_1y_2}(z) = (1-z)^{-1} - 7(1-z)^{-2} + 9(1-z)^{-3} - 13(1-z)^{-4} - 18(1-z)^{-5}$$

Example (of C^-_w and B^-_w)

w	C^-_w	B^-_w	w	C^-_w	B^-_w
y_n	$\frac{1}{n+1}$	$n!$	$y_m y_n$	$\frac{1}{(n+1)(m+n+2)}$	$n!m!\binom{m+n+1}{n+1}$
y_0^2	$\frac{1}{2}$	1	$y_2y_2y_3$	$\frac{1}{280}$	12960
y_0^n	$\frac{1}{n!}$	1	$y_2y_{10}y_1^2$	$\frac{1}{2160}$	9686476800
y_1^2	$\frac{1}{8}$	3	$y_2^2y_4y_3y_{11}$	$\frac{1}{2612736}$	4167611825465088000000

Second global renormalization of divergent polyzetas

$$C^- := \sum_{w \in Y_0^*} C_w^- \; w, \quad \mathrm{H}^- := \sum_{w \in Y_0^*} \mathrm{H}_w^- \; w. \quad \mathrm{L}^- := \sum_{w \in Y_0^*} \mathrm{Li}_w^- \; w.$$

Theorem (Bùi, Duchamp, HNM, Ngô, 2014)

H^- and C^- are group-like respectively for $\Delta_{⊎}$ and $\Delta_{ш}$, and

$$\lim_{z \to 1} \Lambda^{\odot -1}((1-z)^{-1}) \odot \mathrm{Li}^-(z) = \lim_{N \to +\infty} \Upsilon^{\odot -1}(N) \odot \mathrm{H}^-(N) = C^-,$$

where $\Upsilon(t) := \sum_{w \in Y_0^*} t^{(w)+|w|} w$ *and* $\Lambda(t) := \sum_{w \in Y_0^*} ((w)+|w|)! t^{(w)+|w|} w.$

Theorem (Section orbit, Bùi, Duchamp, HNM, Ngô, 2014)

1. *The following maps are surjective morphisms of algebras*

$$\begin{array}{lcccl} \mathrm{H}_\bullet^- & : & (\mathbb{Q}\langle Y_0 \rangle, ⊎) & \longrightarrow & (\mathbb{Q}\{\mathrm{H}_w^-\}_{w \in Y_0^*}, .), \quad w \longmapsto \mathrm{H}_w^-, \\ \mathrm{Li}_\bullet^- & : & (\mathbb{Q}\langle Y_0 \rangle, \top) & \longrightarrow & (\mathbb{Q}\{\mathrm{Li}_w^-\}_{w \in Y_0^*}, .), \quad w \longmapsto \mathrm{Li}_w^-, \end{array}$$

   where $\top$ is a law of algebra in $\mathbb{Q}\langle Y_0 \rangle$ not dualizable.

   Moreover, $\ker \mathrm{H}_\bullet^- = \ker \mathrm{Li}_\bullet^- = \mathbb{Q}\{w - w \top 1_{Y_0^*} | w \in Y_0^*\}$.

2. *Let $\top' : \mathbb{Q}\langle Y_0 \rangle \times \mathbb{Q}\langle Y_0 \rangle \longrightarrow \mathbb{Q}\langle Y_0 \rangle$ be a law such that $\mathrm{Li}_\bullet^-$ is a morphism for $\top'$ and $(1_{Y_0^*} \top' \mathbb{Q}\langle Y_0 \rangle) \cap \ker(\mathrm{Li}_\bullet^-) = \{0\}$.*

   Then $\top' = g \circ \top$, where $g \in GL(\mathbb{Q}\langle Y_0 \rangle)$ such that $\mathrm{Li}_\bullet^- \circ g = \mathrm{Li}_\bullet^-$.

Extension of $\mathrm{Li}_\bullet$ over $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$

Let $\lambda(z)=z/(1-z)$ belongs to the differential ring $\mathcal{C}=\mathbb{C}[z,1/z,1/(1-z)]$ with $\partial_z=d/dz$ and $1_\Omega:\Omega\longrightarrow\mathbb{C}, z\longmapsto 1$, where $\Omega=\mathbb{C}-(]-\infty,0]\cup[1,+\infty[)$.

Let us extend, by linearity and continuity, $\mathrm{Li}_\bullet$ over[7] $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$, via **Lazard**'s elimination (1960), as follows

$$\begin{aligned}S &= \sum_{n\ge0}\langle S|x_0^n\rangle x_0^n+\sum_{k\ge1}\sum_{w\in(x_0^*x_1)^kx_0^*}\langle S|w\rangle w,\\ \mathrm{Li}_S(z) &= \sum_{n\ge0}\langle S|x_0^n\rangle\frac{\log^n(z)}{n!}+\sum_{k\ge1}\sum_{w\in(x_0^*x_1)^kx_0^*}\langle S|w\rangle\,\mathrm{Li}_w(z).\end{aligned}$$

The morphism $\mathrm{Li}_\bullet$ is no longer injective over $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ but $\{\mathrm{Li}_w\}_{w\in X^*}$ are still linearly independant over $\mathcal{C}$ (**HNM**, 2003).

Example

i. $1_\Omega=\mathrm{Li}_{1_{X^*}}=\mathrm{Li}_{x_1^*-x_0^*\,ш\,x_1^*}$.

ii. $\lambda=\mathrm{Li}_{(x_0+x_1)^*}=\mathrm{Li}_{x_0^*\,ш\,x_1^*}=\mathrm{Li}_{x_1^*x_1}$.

iii. $\mathcal{C}=\mathbb{C}[(1-z)^{-1}][z,z^{-1}]=\mathbb{C}[\mathrm{Li}_{x_1^*}][\mathrm{Li}_{x_0^*},\mathrm{Li}_{(-x_0)^*}]$.

iv. $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}=\mathbb{C}[\mathrm{Li}_{x_1^*}][\mathrm{Li}_{x_0^*},\mathrm{Li}_{(-x_0)^*}][\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}]$.

---

[7] $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ = the closure by $\{+,\mathrm{conc},{}^*\}$ of $\mathbb{C}\langle X\rangle$, where, $\forall S\in\mathbb{C}\langle\langle X\rangle\rangle$ s.t. $\langle S|1_{X^*}\rangle=0$, one has $S^*=\sum_{k\ge0}S^k$. $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ is also shuffle closed.

Bi-integro-differential algebra $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ (1/2)

Let us consider the following operators over $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$

$$\theta_1 = (1-z)d/dz, \qquad \theta_0 = zd/dz,$$
$$\iota_1(f) = \int_0^z f(t)\omega_1(t), \qquad \iota_0(f) = \int_{z_0}^z f(s)\omega_0(s),$$

$z_0 = k$ for $f \in B_k$; $k = 0, 1$ with $B = B_0 \sqcup B_1$ is a $\mathbb{C}$-basis adapted to **Lazard's elimination** as previously.

Proposition

1. *$\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ is closed under the action of $\{\theta_0, \theta_1, \iota_0, \iota_1\}$.*
2. *The operators $\{\theta_0, \theta_1, \iota_0, \iota_1\}$ satisfy in particular,*
$$\theta_1 + \theta_0 = [\theta_1, \theta_0] = \partial_z \quad \textit{and} \quad \forall k = 0, 1, \theta_k\iota_k = \mathrm{Id},$$
$$[\theta_0\iota_1, \theta_1\iota_0] = 0 \quad \textit{and} \quad (\theta_0\iota_1)(\theta_1\iota_0) = (\theta_1\iota_0)(\theta_0\iota_1) = \mathrm{Id}.$$
3. *$\theta_0\iota_1$ and $\theta_1\iota_0$ are scalar operators within $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$, respectively with eigenvalues λ and $1/\lambda$, i.e. $(\theta_0\iota_1)f = \lambda f$ and $(\theta_1\iota_0)f = f/\lambda$.*
4. *Let $w = y_{s_1}\ldots y_{s_r} \in Y^*$ (then $\pi_X(w) = x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1$) and $u = y_{t_1}\ldots y_{t_r} \in Y_0^*$. The functions $\mathrm{Li}_w, \mathrm{Li}_u^-$ satisfy*
$$\mathrm{Li}_w = (\iota_0^{s_1-1}\iota_1\ldots\iota_0^{s_r-1}\iota_1)1_\Omega \quad \textit{and} \quad \mathrm{Li}_u^- = (\theta_0^{t_1+1}\iota_1\ldots\theta_0^{t_r+1}\iota_1)1_\Omega,$$
$$\iota_0\,\mathrm{Li}_{\pi_X(w)} = \mathrm{Li}_{x_0\pi_X(w)} \quad \textit{and} \quad \iota_1\,\mathrm{Li}_w = \mathrm{Li}_{x_1\pi_X(w)},$$
$$\theta_0\,\mathrm{Li}_{x_0\pi_X(w)} = \mathrm{Li}_{\pi_X(w)} \quad \textit{and} \quad \theta_1\,\mathrm{Li}_{x_1\pi_X(w)} = \mathrm{Li}_{\pi_X(w)},$$
$$\theta_0\,\mathrm{Li}_{x_1\pi_X(w)} = \lambda\,\mathrm{Li}_{\pi_X(w)} \quad \textit{and} \quad \theta_0\,\mathrm{Li}_{x_1\pi_X(w)} = \mathrm{Li}_{\pi_X(w)}/\lambda.$$

Bi-integro-differential algebra $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ (2/2)

Let $\Im$ and Θ be the $\mathbb{C}$-algebra morphisms $\mathbb{C}\langle X\rangle \to \mathrm{End}_{\mathbb{C}}(\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*})$ defined by, for $v \in X^*, x_i \in X$, $\Im(vx_i) = \Im(v)\iota_i$ and $\Theta(vx_i) = \Theta(v)\theta_i$, and $\Im(1_{X^*}) = \Theta(1_{X^*}) = \mathrm{Id}$.

For any $n \geq 0$ and $u \in X^*, f, g \in \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$, one has

$$\begin{aligned} \partial_z^n &= \sum_{w\in X^n} \mu \circ (\Theta \otimes \Theta)[\Delta_{ш}(w)], \\ \Theta(u)(fg) &= \mu \circ (\Theta \otimes \Theta)[\Delta_{ш}(u)] \circ (f \otimes g). \end{aligned}$$

Theorem (extension of $\mathrm{Li}_\bullet$)

$$\begin{aligned} \mathrm{Li}_\bullet : (\mathbb{C}[x_0^*]ш\mathbb{C}[(-x_0)^*]ш\mathbb{C}[x_1^*]ш\mathbb{C}\langle X\rangle, ш, 1_{X^*}) &\longrightarrow (\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, \times, 1), \\ T &\longmapsto \Im(T)1_\Omega. \end{aligned}$$

$\mathrm{Li}_\bullet$ *is surjective and* $\ker \mathrm{Li}_\bullet$ *is the ideal generated by* $x_0^* ш x_1^* - x_1^* + 1$.

Theorem (derivations on $(\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, \times, 1)$)

The morphism of $\mathbb{C}$-AAU Θ maps $\mathcal{L}ie_{\mathbb{C}}\langle X\rangle$ to $\mathfrak{Der}(\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, \times, 1))$, its image is the Lie algebra generated by θ_0, θ_1.

Because, for $P, Q \in \mathbb{C}[x_0^*]ш\mathbb{C}[(-x_0)^*]ш\mathbb{C}[x_1^*]ш\mathbb{C}\langle X\rangle$ and $T \in \mathcal{L}ie_{\mathbb{C}}\langle X\rangle$, one has $\mathrm{Li}_{PшQ} = \mathrm{Li}_P \mathrm{Li}_Q$ and $\Theta(T)\,\mathrm{Li}_{PшQ} = \mathrm{Li}_{(PшQ)\triangleleft T}$ and then

$$\begin{aligned} \Theta(T)(\mathrm{Li}_P \mathrm{Li}_Q) &= \mathrm{Li}_{(PшQ)\triangleleft T} = \mathrm{Li}_{(P\triangleleft T)шQ + Pш(Q\triangleleft T)} \\ &= \mathrm{Li}_{(P\triangleleft T)шQ} + \mathrm{Li}_{Pш(Q\triangleleft T)} = (\Theta(T)\,\mathrm{Li}_P)\,\mathrm{Li}_Q + \mathrm{Li}_P(\Theta(T)\,\mathrm{Li}_Q). \end{aligned}$$

Actions of $\{\theta_0,\theta_1,\iota_0,\iota_1\}$ over $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ (1/2)

$$\forall u\in Y^*, \mathrm{Li}^-_{y_{s_1}u} = \theta_0^{s_1}(\theta_0\iota_1)\,\mathrm{Li}^-_u = \theta_0^{s_1}(\lambda\,\mathrm{Li}^-_u) = \sum_{k_1=0}^{s_1}\binom{s_1}{k_1}(\theta_0^{k_1}\lambda)(\theta_0^{s_1-k_1}\,\mathrm{Li}^-_u).$$

$$\Rightarrow \mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}} = \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\cdots\sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\cdots\binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r}(\theta_0^{k_1}\lambda)(\theta_0^{k_2}\lambda)\ldots(\theta_0^{k_r}\lambda),$$

$$\theta_0^{k_i}\lambda(z) = \begin{cases} z(1-z)^{-1}, & \text{if} \quad k_i=0,\\ (1-z)^{-1}\sum_{j=1}^{k_i}S_2(k_i,j)j!(z(1-z)^{-1})^j, & \text{if} \quad k_i>0.\end{cases}$$

Hence, $\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}} = \mathrm{Li}_T = \Im(T)1_\Omega$, where $T\in\mathbb{C}[x_0^*]ш\mathbb{C}[x_1^*]$ given by

$$T = \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\cdots\sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\cdots\binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r}T_{k_1}ш\ldotsш T_{k_r},$$

$$T_{k_i} = \begin{cases} x_0^*шx_1^*, & \text{if} \quad k_i=0,\\ x_1^*ш\sum_{j=1}^{k_i}S_2(k_i,j)j!(x_0^*шx_1^*)^{шj}, & \text{if} \quad k_i>0.\end{cases}$$

Actions of $\{\theta_0, \theta_1, \iota_0, \iota_1\}$ over $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ (2/2)

Due to surjectivity of
$\mathrm{Li}_\bullet : (\mathbb{C}[x_0^*] ш \mathbb{C}[(-x_0)^*] ш \mathbb{C}[x_1^*] ш \mathbb{C}\langle X\rangle, ш, 1_{X^*}) \longrightarrow\!\!\!\!\rightarrow (\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, \times, 1)$,
one also has $\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}} = \mathrm{Li}_F = \Im(F)1_\Omega$, where $F \in \mathbb{C}[x_1^*]$ given by

$$F = \sum_{k_1=0}^{s_1} \sum_{k_2=0}^{s_1+s_2-k_1} \ldots \sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}} \binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\ldots \binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r} F_{k_1} ш \ldots ш F_{k_r},$$

$$F_{k_i} = \begin{cases} x_1^* - 1, & \text{if } \quad k_i = 0, \\ x_1^* ш \sum_{j=1}^{k_i} S_2(k_i,j) j! (x_1^* - 1)^{ш j}, & \text{if } \quad k_i > 0. \end{cases}$$

Conversely, for any $k \in \mathbb{N}_+$, one has

$$\Im((x_1^*)^{ш k})1_\Omega = \frac{1}{(1-z)^k} = (-1)^k(\mathrm{Li}^-_{y_0}(z) - 1) + \sum_{j=2}^{k} \frac{(-1)^{k+j} S_1(k,j)}{(k-1)!} \mathrm{Li}^-_{y_{j-1}}(z).$$

Corollary

$$\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*} \supsetneq \mathbb{C}[1/(1-z)]\{\mathrm{Li}_w\}_{w\in X^*} = \Im(\mathbb{C}[x_1^*] ш \mathbb{C}\langle X\rangle)1_\Omega$$
$$= \mathrm{span}_{\mathbb{C}} \left\{ \sum_{n_1 > \ldots > n_r > 0} n_1^{s_1} \ldots n_r^{s_r} z^{n_1} \right\}_{(s_1,\ldots,s_r)\in\mathbb{Z}^r, r\in\mathbb{N}_+}.$$

SOME CONSEQUENCES
ON STRUCTURE OF POLYZETAS

Homogenous polynomials relations among local coordinates

$$Z_\gamma = \Gamma(y_1+1)\pi_Y Z_{\text{III}}$$

	Relations among $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Relations among $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) = \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) = \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) = \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) = \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) = \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) = 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) = -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) = -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) = \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) = \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) = \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) = \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) = \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) = \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) = \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) = \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) = -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) = \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) = 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) = \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) = \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) = -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) = \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) = \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) = \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) = \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$

Noetherian rewriting system & irreducible coordinates

$$Z_\gamma = \Gamma(y_1+1)\pi_Y Z_{ш}$$

	Rewriting among $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Rewriting among $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) \to \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) \to \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) \to \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) \to \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) \to 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) \to -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) \to -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) \to \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) \to \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) \to \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) \to \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) \to \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) \to \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) \to \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) \to \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) \to -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) \to \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) \to 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) \to \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) \to -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) \to \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) \to \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) \to \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) \to \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$

Examples of irreducible local coordinates by computer

$$\mathcal{Z}_{irr}^{\leq 12}(Y) := \{\zeta(\Sigma_{y_2}), \zeta(\Sigma_{y_3}), \zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_3 y_1^5}), \zeta(\Sigma_{y_9}), \zeta(\Sigma_{y_3 y_1^7}), \zeta(\Sigma_{y_{11}}), \zeta(\Sigma_{y_2 y_1^9}), \zeta(\Sigma_{y_3 y_1^9}), \zeta(\Sigma_{y_2^2 y_1^8})\}.$$

$$\mathcal{Z}_{irr}^{\leq 12}(X) := \{\zeta(S_{x_0 x_1}), \zeta(S_{x_0^2 x_1}), \zeta(S_{x_0^4 x_1}), \zeta(S_{x_0^6 x_1}), \zeta(S_{x_0 x_1^2 x_0 x_1^4}), \zeta(S_{x_0^8 x_1}), \zeta(S_{x_0 x_1^2 x_0 x_1^6}), \zeta(S_{x_0^{10} x_1}), \zeta(S_{x_0 x_1^3 x_0 x_1^7}), \zeta(S_{x_0 x_1^2 x_0 x_1^8}), \zeta(S_{x_0 x_1^4 x_0 x_1^6})\}$$

Theorem (HNM, 2009)

$\mathcal{Z}_{irr}^{\leq 12}(Y)$ (resp. $\mathcal{Z}_{irr}^{\leq 12}(X)$) constitutes a system of local coordinates. Their elements are algebraically independant if and only if the Zagier's dimension conjecture holds up to weight 12.

$$\mathcal{L}_{irr}^{\leq 12}(Y) := (\mathbb{Q}[\Sigma_{y_2}, \Sigma_{y_3}, \Sigma_{y_5}, \Sigma_{y_7}, \Sigma_{y_3 y_1^5}, \Sigma_{y_9} \Sigma_{y_3 y_1^7}, \Sigma_{y_{11}}, \Sigma_{y_2 y_1^9}, \Sigma_{y_3 y_1^9}, \Sigma_{y_2^2 y_1^8}], \,⧢\!\!\!\!+\,, 1_{Y^*}).$$

$$\mathcal{L}_{irr}^{\leq 12}(X) := (\mathbb{Q}[S_{x_0 x_1}, S_{x_0^2 x_1}, S_{x_0^4 x_1}, S_{x_0^6 x_1}, S_{x_0 x_1^2 x_0 x_1^4}, S_{x_0^8 x_1}, S_{x_0 x_1^2 x_0 x_1^6}, S_{x_0^{10} x_1}, S_{x_0 x_1^3 x_0 x_1^7}, S_{x_0 x_1^2 x_0 x_1^8}, S_{x_0 x_1^4 x_0 x_1^6}], ⧢, 1_{X^*}).$$

$$\mathcal{L}_{irr}^{\infty}(Y) := \bigcup_{p \geq 2} \mathcal{L}_{irr}^{\leq p}(Y) \quad \text{and} \quad \mathcal{L}_{irr}^{\infty}(X) := \bigcup_{p \geq 2} \mathcal{L}_{irr}^{\leq p}(X).$$

Homogenous polynomials generating $\ker\zeta$

	$\{Q_l\}_{l\in\mathcal{L}ynY-\{y_1\}}$	$\{Q_l\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1} - \frac{3}{2}\Sigma_{y_3}) = 0$	$\zeta(S_{x_0x_1^2} - S_{x_0^2x_1}) = 0$
4	$\zeta(\Sigma_{y_4} - \frac{2}{5}\Sigma_{y_2}^{\uplus 2}) = 0$	$\zeta(S_{x_0^3x_1} - \frac{2}{5}S_{x_0x_1}^{\text{Ш}2}) = 0$
	$\zeta(\Sigma_{y_3y_1} - \frac{3}{10}\Sigma_{y_2}^{\uplus 2}) = 0$	$\zeta(S_{x_0^2x_1^2} - \frac{1}{10}S_{x_0x_1}^{\text{Ш}2}) = 0$
	$\zeta(\Sigma_{y_2y_1^2} - \frac{2}{3}\Sigma_{y_2}^{\uplus 2}) = 0$	$\zeta(S_{x_0x_1^3} - \frac{2}{5}S_{x_0x_1}^{\text{Ш}2}) = 0$
5	$\zeta(\Sigma_{y_3y_2} - 3\Sigma_{y_3} \uplus \Sigma_{y_2} - 5\Sigma_{y_5}) = 0$	$\zeta(S_{x_0^3x_1^2} - S_{x_0^2x_1}\,\text{Ш}\,S_{x_0x_1} + 2S_{x_0^4x_1}) = 0$
	$\zeta(\Sigma_{y_4y_1} - \Sigma_{y_3} \uplus \Sigma_{y_2}) + \frac{5}{2}\Sigma_{y_5}) = 0$	$\zeta(S_{x_0^2x_1x_0x_1} - \frac{3}{2}S_{x_0^4x_1} + S_{x_0^2x_1}\,\text{Ш}\,S_{x_0x_1}) = 0$
	$\zeta(\Sigma_{y_2^2y_1} - \frac{3}{2}\Sigma_{y_3} \uplus \Sigma_{y_2} - \frac{25}{12}\Sigma_{y_5}) = 0$	$\zeta(S_{x_0^2x_1^3} - S_{x_0^2x_1}\,\text{Ш}\,S_{x_0x_1} + 2S_{x_0^4x_1}) = 0$
	$\zeta(\Sigma_{y_3y_1^2} - \frac{5}{12}\Sigma_{y_5}) = 0$	$\zeta(S_{x_0x_1x_0x_1^2} - \frac{1}{2}S_{x_0^4x_1}) = 0$
	$\zeta(\Sigma_{y_2y_1^3} - \frac{1}{4}\Sigma_{y_3} \uplus \Sigma_{y_2}) + \frac{5}{4}\Sigma_{y_5}) = 0$	$\zeta(S_{x_0x_1^4} - S_{x_0^4x_1}) = 0$
6	$\zeta(\Sigma_{y_6} - \frac{8}{35}\Sigma_{y_2}^{\uplus 3}) = 0$	$\zeta(S_{x_0^5x_1} - \frac{8}{35}S_{x_0x_1}^{\text{Ш}3}) = 0$
	$\zeta(\Sigma_{y_4y_2} - \Sigma_{y_3}^{\uplus 2} - \frac{4}{21}\Sigma_{y_2}^{\uplus 3}) = 0$	$\zeta(S_{x_0^4x_1^2} - \frac{6}{35}S_{x_0x_1}^{\text{Ш}3} - \frac{1}{2}S_{x_0^2x_1}^{\text{Ш}2}) = 0$
	$\zeta(\Sigma_{y_5y_1} - \frac{2}{7}\Sigma_{y_2}^{\uplus 3} - \frac{1}{2}\Sigma_{y_3}^{\uplus 2}) = 0$	$\zeta(S_{x_0^3x_1x_0x_1} - \frac{4}{105}S_{x_0x_1}^{\text{Ш}3}) = 0$
	$\zeta(\Sigma_{y_3y_1y_2} - \frac{17}{30}\Sigma_{y_2}^{\uplus 3} + \frac{9}{4}\Sigma_{y_3}^{\uplus 2}) = 0$	$\zeta(S_{x_0^3x_1^3} - \frac{23}{70}S_{x_0x_1}^{\text{Ш}3} - S_{x_0^2x_1}^{\text{Ш}2}) = 0$
	$\zeta(\Sigma_{y_3y_2y_1} - 3\Sigma_{y_3}^{\uplus 2} - \frac{9}{10}\Sigma_{y_2}^{\uplus 3}) = 0$	$\zeta(S_{x_0^2x_1x_0x_1^2} - \frac{2}{105}S_{x_0x_1}^{\text{Ш}3}) = 0$
	$\zeta(\Sigma_{y_4y_1^2} - \frac{3}{10}\Sigma_{y_2}^{\uplus 2} - \frac{3}{4}\Sigma_{y_3}^{\uplus 2}) = 0$	$\zeta(S_{x_0^2x_1^2x_0x_1} - \frac{89}{210}S_{x_0x_1}^{\text{Ш}3} + \frac{3}{2}S_{x_0^2x_1}^{\text{Ш}2}) = 0$
	$\zeta(\Sigma_{y_2^2y_1^2} - \frac{11}{63}\Sigma_{y_2}^{\uplus 2} - \frac{1}{4}\Sigma_{y_3}^{\uplus 2}) = 0$	$\zeta(S_{x_0^2x_1^4} - \frac{6}{35}S_{x_0x_1}^{\text{Ш}3} - \frac{1}{2}S_{x_0^2x_1}^{\text{Ш}2}) = 0$
	$\zeta(\Sigma_{y_3y_1^3} - \frac{1}{21}\Sigma_{y_2}^{\uplus 3}) = 0$	$\zeta(S_{x_0x_1x_0x_1^3} - \frac{8}{21}S_{x_0x_1}^{\text{Ш}3} - S_{x_0^2x_1}^{\text{Ш}2}) = 0$
	$\zeta(\Sigma_{y_2y_1^4} - \frac{17}{50}\Sigma_{y_2}^{\uplus 3} + \frac{3}{16}\Sigma_{y_3}^{\uplus 2}) = 0$	$\zeta(S_{x_0x_1^5} - \frac{8}{35}S_{x_0x_1}^{\text{Ш}3}) = 0$

$\mathcal{R}_Y := (\mathbb{Q}\{Q_l\}_{l\in\mathcal{L}ynY-\{y_1\}}, \uplus, 1_{Y^*})$ and $\mathcal{R}_X := (\mathbb{Q}\{Q_l\}_{l\in\mathcal{L}ynX-X}, \text{Ш}, 1_{X^*})$

Structure of polyzetas

$$\mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1 = \mathcal{R}_X \oplus \mathcal{L}_{irr}^{\infty}(X) \quad \text{and} \quad \mathcal{R}_X \subset \ker\zeta,$$
$$\mathbb{Q}1_{Y^*} \oplus (Y - \{y_1\})\mathbb{Q}\langle Y\rangle = \mathcal{R}_Y \oplus \mathcal{L}_{irr}^{\infty}(Y) \quad \text{and} \quad \mathcal{R}_Y \subset \ker\zeta.$$
$$\zeta : \begin{matrix} (\mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1, ш, 1_{X^*}) \\ (\mathbb{Q}1_{Y^*} \oplus (Y - \{y_1\})\mathbb{Q}\langle Y\rangle, ⧢, 1_{Y^*}) \end{matrix} \longrightarrow\!\!\!\!\rightarrow (\mathcal{Z}, .),$$
$$\begin{matrix} x_0x_1^{r_1-1}\ldots x_0x_1^{r_k-1} \\ y_{s_1}\ldots y_{s_k} \end{matrix} \longmapsto \sum_{n_1>\ldots>n_k>0} \frac{1}{n_1^{s_1}\ldots n_k^{s_k}}.$$

Since $\mathrm{Im}\,\zeta = \mathcal{Z}$ then restricted on $\mathcal{L}_{irr}^{\infty}(X)$ (resp. $\mathcal{L}_{irr}^{\infty}(Y)$) ζ is injective (**HNM**, 2009). Hence, $\ker\zeta = \mathcal{R}_X$ (resp. $\ker\zeta = \mathcal{R}_Y$) generated by homogenous polynomials and $\mathrm{Im}\,\zeta \cong \mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1/\ker\zeta \cong \mathcal{L}_{irr}^{\infty}(X)$ (resp. $\mathrm{Im}\,\zeta \cong \mathbb{Q}1_{Y^*} \oplus (Y - \{y_1\})\mathbb{Q}\langle Y\rangle/\ker\zeta \cong \mathcal{L}_{irr}^{\infty}(Y)$.

Theorem (HNM, 2009)

$$\mathcal{Z} = \mathbb{Q} \oplus \bigoplus_{p\geq 2} \mathcal{Z}_p, \textit{ where } \mathcal{Z}_p = \mathrm{span}_{\mathbb{Q}}\{\zeta(w) \mid w \in x_0X^*x_1, |w| = p\}.$$

Let $P \in \mathbb{Q}\langle X\rangle$ homogenous of degree n. Suppose $\xi = \zeta(P)$ satisfies $\xi^n + a_{n-1}\xi^{n-1} + \ldots = 0$, in which each monomial is of different weight because $\mathcal{Z}_{p_1}\mathcal{Z}_{p_2} \subset \mathcal{Z}_{p_1+p_2}$. Then ξ is a transcendental number over $\mathbb{Q}$. Since $\{S_l\}_{l\in\mathcal{L}ynY}$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$) are homogenous in weight then

Corollary

Any irreducible polyzeta is a transcendental number over $\mathbb{Q}$.

Discussion and conclusion

In all cases $(A\langle Y\rangle, ш_\varphi, 1, \Delta_{\mathrm{conc}}, \epsilon)$ is a Hopf algebra (see recent literature), Hoffman's exponential offers an isomorphism between it and shuffle bialgebra ($\varphi \equiv 0$).

Our case is the dual discussion. Even if φ is dualizable (and we have dual structures) it may happen that $(A\langle Y\rangle, \mathrm{conc}, 1, \Delta_{ш_\varphi}, \epsilon)$ is NOT a Hopf algebra and that Hoffman's correspondence had NO counterpart (see in the case of the infiltration products of Chen, Fox & Lyndon, 1958).

In the case when φ is moderate, we get at once enveloping algebras and the Lie algebra of primitive elements admits effective bases for which the polynomiality of the dual basis is guaranteed (even in the non-graded cases). An isomorphism with $(A\langle Y\rangle, \mathrm{conc}, 1, \Delta_{ш}, \epsilon)$ can be effectively constructed by sending every letter to its image through the $\log_*$.

Here, we have

1. studied generic φ (or mixed) deformations of the shuffle algebras
2. discussed the possibilities of dualization w.r.t. the growth of φ
3. applied these results to (global) multiplicative renormalization of polyzetas with (positive or negative) indices

THANK YOU FOR YOUR ATTENTION