Path Integrals, asymptotic expansions and Zeta determinants

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The heat equation

Let $L = \nabla^*\nabla$ be a Laplace type operator, acting on sections of a vector bundle $\mathcal{V}$ on a compact n-dimensional Riemannian manifold M.

The *heat* or *diffusion equation*

$$\frac{\partial}{\partial t}u(t,x) + Lu(t,x) = 0, \qquad u(0,x) = u_0(x)$$

has a unique solution for given initial data $u_0 \in L^2(M, \mathcal{V})$, and it is given by

$$u(t,x) = \int_M p_t^L(x,y)u(y)\mathrm{d}y, \quad t > 0$$

where $p_t^L \in C^\infty(M \times M, \mathcal{V} \boxtimes \mathcal{V}^*)$ is the *heat kernel* of L.

The heat kernel as a path integral

By physicist's reasoning, the heat kernel can be written as the "sum over all histories", weighted with their probability.

$$p_t^L(x,y) \overset{\text{formally}}{=} (4\pi t)^{-n/2} \fint_{\substack{\text{paths} \\ x\mapsto y}} \exp\Big(- \frac{1}{4t}\int_0^1 \big|\dot\gamma(s)\big|^2 \mathrm{d}s \Big) [\gamma\|_0^1]^{-1} \, \mathcal{D}\gamma$$

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The slash in the integral sign denotes division by the normalization

$$Z = (4\pi t)^{N/2}, \qquad N = \text{dimension of path space}$$

$\mathcal{D}\gamma$ denotes the Riemannian volume measure of the space of paths.

The heat kernel as a path integral

$$p_t^L(x,y) \overset{\text{formally}}{=} (4\pi t)^{-n/2} \fint_{\substack{\text{paths} \\ x \mapsto y}} e^{-E(\gamma)/2t} \, [\gamma\|_0^1]^{-1} \, \mathcal{D}\gamma$$

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Theorem

We have

$$p_t^L(x,y) = \int_{C_{xy}(M)} [\gamma\|_0^1]^{-1}\, \mathrm{d}\mathbb{W}^{xy;t}(\gamma).$$

In the theorem, $[\gamma\|_0^1]^{-1}$ is the stochastic parallel transport in $\mathcal{V}$, $\mathbb{W}^{xy;t}$ is a conditional Wiener measure and

$$C_{xy;t}(M) = \{\gamma \in C([0,t],M) \mid \gamma(0)=x, \gamma(1)=y\}$$

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Theorem (L. '15)

We have

$$p_t^L(x,y) = \lim_{|\tau|\to 0} \int_{H_{xy;\tau}(M)} e^{-E(\gamma)/2t} \, [\gamma\|_0^1]^{-1} \, \mathrm{d}^{\Sigma\text{-}H^1}\gamma$$

where the limit goes over any sequence of partitions $\tau = \{0 = \tau_0 < \tau_1 < \cdots < \tau_N = 1\}$ *of the interval* $[0,1]$, *the mesh of which tends to zero.*

In the theorem,

$$H_{xy;\tau}(M) = \left\{\gamma \in H_{xy}(M) \mid \gamma|_{[\tau_{j-1},\tau_j]} \text{ is a geodesic } \forall j\right\}$$

The space of finite energy paths

An important path space is the space of finite energy paths

$$H_{xy}(M) := \{\gamma \in H^1([0,t], M) \mid \gamma(0) = x, \gamma(1) = y\}.$$

This is an infinite-dimensional Hilbert manifold with the Riemannian metric

$$(X, Y)_{H^1} := \int_0^1 \langle \nabla_s X, \nabla_s Y \rangle \mathrm{d}s, \qquad X, Y \in T_\gamma H_{xy}(M).$$

It is the "Cameron-Martin-manifold" corresponding to the Wiener measure.

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- The heat kernel has an asymptotic expansion

$$p_t^L(x,y) \sim \mathrm{e}_t(x,y)\sum_{j=0}^{\infty} t^j \Phi_j(x,y), \qquad \mathrm{e}_t(x,y) = \frac{e^{-d(x,y)^2/4t}}{(4\pi t)^{n/2}}$$

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- The path integral has a *formal* Laplace expansion.

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Goal

Compare the two asymptotic expansions.

In this talk: The lowest order term.

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Laplace's Expansion

Let Ω be a finite-dimensional Riemannian manifold. If $\phi : \Omega \longrightarrow \mathbb{R}$ has the unique non-degenerate minimum x_0, then

$$\int_\Omega e^{-\phi(x)/2t} a(x)\,\mathrm{d}x \;\sim\; e^{-\phi(x_0)/2t} \frac{a(x_0)}{\det\bigl(\nabla^2\phi|_{x_0}\bigr)^{1/2}}$$

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Formal conclusion

If there is a unique shortest geodesic γ_{xy} connecting x to y, then *formally*

$$\int_{H_{xy}(M)} e^{-E(\gamma)/2t} [\gamma\|_0^t]^{-1} \mathcal{D}\gamma \;\sim\; e^{-d(x,y)^2/4t} \frac{[\gamma_{xy}\|_0^1]^{-1}}{\det\left(\nabla^2 E|_{\gamma_{xy}}\right)^{1/2}}$$

since $E(\gamma_{xy}) = d(x,y)^2/2$.

A formal proof

Taylor expand $E(\gamma) = E(\gamma_{xy}) + \frac{1}{2}\nabla^2 E|_{\gamma_{xy}}[X, X] + O(|X|^3)$, where X is the vector field with $\exp_{\gamma_{xy}}(X) = \gamma$.

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$$\begin{aligned} &\fint \exp\left(-\frac{E(\gamma)}{2t}\right)\mathcal{D}\gamma \\ &\qquad = \fint \exp\left(-\frac{1}{2t}\left(E(\gamma_{xy}) + \frac{1}{2}\nabla^2 E|_{\gamma_{xy}}[X, X] + O(|X|^3)\right)\right)\mathcal{D}X \end{aligned}$$

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substitute $X \mapsto t^{1/2}X$ gives

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$$\sim \; e^{-E(\gamma_{xy})/2t} \det\left(\nabla^2 E|_{\gamma_{xy}}\right)^{-1/2}$$

Remark

At a geodesic γ, the Hessian of the energy is given by

$$\nabla^2 E|_\gamma[X, Y] = \int_0^1 \big\langle \nabla_s X, \nabla_s Y \big\rangle \mathrm{d}s + \int_0^1 \big\langle\, R\big(\dot\gamma(s), X(s)\big)\dot\gamma(s)\,, Y(s)\big\rangle \mathrm{d}s$$

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Hence

$$\det\big(\nabla^2 E|_\gamma\big) = \det\big(P^{-1}(P+\mathcal{R}_\gamma)\big) = \det\big(\mathrm{id} + \underbrace{P^{-1}\mathcal{R}_\gamma}_{\text{trace-class}}\big).$$

Theorem (L. '15)

For points $x, y \in M$ such that there is a unique minimal geodesic connecting x to y, we have

$$\frac{p_t^L(x,y)}{\mathrm{e}_t(x,y)} \sim \frac{[\gamma_{xy}\|_0^1]^{-1}}{\det\big(\nabla^2 E|_{\gamma_{xy}}\big)^{1/2}}$$

where $\mathrm{e}_t(x,y)$ is the "Euclidean heat kernel".

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Corollary

The Jacobian of the exponential map is a Fredholm determinant,

$$\det\big(d\exp_x|_{\dot\gamma_{xy}(0)}\big) = \det\big(\nabla^2 E|_{\gamma_{xy}}\big)$$

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Proof of Corollary. It is well-known that

$$\frac{p_t^L(x,y)}{\mathrm{e}_t(x,y)} \sim \frac{[\gamma_{xy}\|_0^1]^{-1}}{\det\left(d\exp_x|_{\dot\gamma_{xy}(0)}\right)^{1/2}}.$$

□

This can be proved by finite-dimensional approximation, but one has to make precise error estimates for the approximation.

Theorem (L. '15)

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$$p_t^L(x,y) = \lim_{|\tau|\to 0} \int_{H_{xy;\tau}(M)} e^{-E(\gamma)/2t} \, [\gamma\|_0^1]^{-1} \, \mathrm{d}^{\Sigma\text{-}H^1}\gamma$$

where the limit goes over any sequence of partitions $\tau = \{0 = \tau_0 < \tau_1 < \cdots < \tau_N = 1\}$ *of the interval* $[0,1]$*, the mesh of which tends to zero.*

In the physics literature, one reads

$$\int e^{-E(\gamma)/2t}\mathcal{D}\gamma \;\sim\; e^{-E(\gamma_{xy})/2t}\int \exp\left(-\frac{1}{4}\nabla^2 E|_{\gamma_{xy}}[X,X]\right)\mathcal{D}X$$

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$$\begin{aligned}\fint e^{-E(\gamma)/2t}\mathcal{D}\gamma &\sim e^{-E(\gamma_{xy})/2t}\fint \exp\left(-\frac{1}{4}\nabla^2 E|_{\gamma_{xy}}[X,X]\right)\mathcal{D}X \\ &\sim e^{-E(\gamma_{xy})/2t}\fint \exp\left(-\frac{1}{4}\big(X,(P+\mathcal{R}_{\gamma_{xy}})X\big)_{L^2}\right)\mathcal{D}X \\ &\propto e^{-E(\gamma_{xy})/2t}\,\det\nolimits_\zeta\big(P+\mathcal{R}_{\gamma_{xy}}\big)^{-1/2}\end{aligned}$$

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From the english Wikipedia article on "Functional Determinants".

> *"This path integral is only well defined up to some divergent multiplicative constant. To give it a rigorous meaning, it must be divided by another functional determinant, thus effectively cancelling the problematic constants."*

Zeta determinant

The zeta function of a *zeta-admissible* operator P is defined by

$$\zeta_P(z) = \sum_{\lambda \neq 0} \lambda^{-z}, \qquad \mathrm{Re}(z) \gg 0.$$

where the sum goes over its eigenvalues.

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$$\zeta_P'(z) = -\sum_{\lambda \neq 0} \log(\lambda) \lambda^{-z} \qquad \Longrightarrow \qquad e^{-\zeta_P'(z)} = \prod_{\lambda \neq 0} \lambda^{\lambda^{-z}}.$$

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This motivates to set

$$\det_\zeta(P) := e^{-\zeta_P'(0)} \overset{\text{formally}}{=} \prod_{\lambda \neq 0} \lambda,$$

defined by analytic continuation.

Zeta determinant and Fredholm determinants

Theorem (Scott '04)

Let P be a closed operator on a Hilbert space H and let $T = \mathrm{id} + W$ with W trace-class. If P is a zeta-admissible, then so is PT and

$$\det_\zeta(PT) = \det_\zeta(P)\det(T).$$

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Corollary

With $P = -\nabla_s^2$, we have

$$\det\big(\nabla^2 E|_\gamma\big) = \det\big(P^{-1}(P + \mathcal{R}_\gamma)\big) = \frac{\det_\zeta(P + \mathcal{R}_\gamma)}{\det_\zeta(P)}.$$

Application

Theorem

For $x, y \in M$ such that there is a unique minimizing geodesic γ_{xy} connecting x and y, we have

$$\frac{p_t^L(x,y)}{\mathrm{e}_t(x,y)} \;\sim\; \det\bigl(\nabla^2 E|_{\gamma_{xy}}\bigr)^{-1/2} = \frac{\det_\zeta(-\nabla_s^2 + \mathcal{R}_{\gamma_{xy}})^{-1/2}}{\det_\zeta(-\nabla_s^2)^{-1/2}}$$

The degenerate case

Let Ω be a finite-dimensional Riemannian manifold. If $\phi : \Omega \longrightarrow \mathbb{R}$ is a function such that $\phi \geq \lambda$ and $C = \phi^{-1}(\lambda)$ is a non-degenerate submanifold of dimension k, then

$$\fint_\Omega e^{-\phi(x)/2t} a(x)\,\mathrm{d}x \ \sim \ (4\pi t)^{-k/2} \int_C \frac{e^{-\lambda/2t} a(x)}{\det\big(\nabla^2 \phi|_{N_x C}\big)^{1/2}} \mathrm{d}x$$

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Formal conclusion

If the set of minimal geodesics $\Gamma_{xy}^{\min} \subseteq H_{xy}(M)$ is a non-degenerate submanifold of dimension k (with respect to the energy functional E), then *formally*

$$\int_{H_{xy}(M)} e^{-E(\gamma)/2t} [\gamma\|_0^t]^{-1} \mathcal{D}\gamma \;\sim\; (4\pi t)^{-k/2} \int_{\Gamma_{xy}^{\min}} \frac{e^{-d(x,y)^2/4t} [\gamma\|_0^1]^{-1}}{\det\big(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}}\big)^{1/2}} \mathrm{d}\gamma.$$

Theorem (L. '15, H^1 picture)

Suppose that $\Gamma_{xy}^{\min} \subseteq H_{xy}(M)$ is a non-degenerate submanifold of dimension k, then

$$\frac{p_t^L(x,y)}{\mathrm{e}_t(x,y)} \sim (4\pi t)^{-k/2} \int_{\Gamma_{xy}^{\min}} [\gamma\|_0^1]^{-1} \frac{1}{\det\left(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}}\right)^{1/2}} \mathrm{d}^{H^1}\gamma.$$

Theorem (L. '15, H^1 picture)

Suppose that $\Gamma_{xy}^{\min} \subseteq H_{xy}(M)$ *is a non-degenerate submanifold of dimension* k*, then*

$$\frac{p_t^L(x,y)}{\mathrm{e}_t(x,y)} \sim (4\pi t)^{-k/2} \int_{\Gamma_{xy}^{\min}} [\gamma\|_0^1]^{-1} \frac{1}{\det\left(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}}\right)^{1/2}} \mathrm{d}^{H^1}\gamma.$$

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Let $e_1, \ldots, e_k$ be an L^2-orthonormal basis of $\Gamma_{xy}^{\min}$. Then we have, setting again $P := -\nabla_s^2$

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Hence in the splitting $T_\gamma H_{xy}(M) = T_\gamma \Gamma_{xy}^{\min} \oplus N_\gamma \Gamma_{xy}^{\min}$

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$$= \left(\frac{\det'_\zeta (P + \mathcal{R}_\gamma)}{\det_\zeta (P)} \right)^{-1/2}$$

Therefore finally

$$\begin{aligned}
&\frac{p_t^L(x,y)}{\mathrm{e}_t(x,y)} \\
&\quad \sim \ (4\pi t)^{-k/2} \int_{\Gamma_{xy}^{\min}} [\gamma\|_0^1]^{-1} \frac{\det\big(d\big[\mathrm{id} : (\Gamma_{xy}^{\min}, L^2) \to (\Gamma_{xy}^{\min}, H^1)\big]\big)}{\det\big(\nabla^2 E|_{N_\gamma \Gamma_{xy}^{\min}}\big)^{1/2}} \mathrm{d}^{L^2}\gamma \\
&\quad = (4\pi t)^{-k/2} \int_{\Gamma_{xy}^{\min}} [\gamma\|_0^1]^{-1} \frac{\det_\zeta(P)^{1/2}}{\det'_\zeta(P + \mathcal{R}_\gamma)^{1/2}} \mathrm{d}^{L^2}\gamma
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□

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For x close to y, we have

$$\frac{p_t^L(x,y)}{\mathrm{e}_t(x,y)} \overset{\text{formally}}{=} \frac{\fint_{H_{xy}(M)} e^{-E(\gamma)/2t}\,\mathcal{D}\gamma}{\fint_{H_{xy}(\mathbb{R}^n)} e^{-E(\gamma)/2t}\,\mathcal{D}\gamma} \sim \frac{\det(\nabla^2 E|_{\gamma_{xy}})^{-1/2}}{\det(\nabla^2 E|_{\gamma_{xy}^{\mathbb{R}^n}})^{-1/2}}$$

or

$$\sim \frac{\det(-\nabla_s^2 + \mathcal{R}_{\gamma_{xy}})^{-1/2}}{\det_\zeta(-\nabla_s^2)}$$

Similar results hold in the degenerate case.

Thank you for your attention!