Renormalisation in regularity structures

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Summary

- Stochastic Partial Differential Equations
- Taylor expansions
- Renormalization groups
- Hopf algebras and co-modules
- Labelled trees and forests
- Feynman diagrams
- ...

Notations

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ we write

$$x^k := \prod_{i=1}^d x_i^{k_i} \in \mathbb{R}.$$

$X = (X_1, \dots, X_d)$ denotes a variable, and X^k the **abstract monomial**

$$X^k := \prod_{i=1}^d X_i^{k_i}.$$

A **monomial** is a function $\Pi_x X^k : \mathbb{R}^d \to \mathbb{R}$

$$\Pi_x X^k(y) := (y - x)^k.$$

Taylor expansions

The Taylor expansion of the function $y \mapsto y^k$ around the fixed base point x is

$$y^k = (y - x + x)^k = \sum_{i=0}^{k} \binom{k}{i} x^{k-i} (y - x)^i$$

$$= \sum_{i=0}^{k} \frac{(y-x)^i}{i!} \left. \frac{\partial^i y^k}{\partial y^i} \right|_{y=x}, \qquad \left. \frac{\partial^i y^k}{\partial y^i} \right|_{y=x} = \frac{k!}{(k-i)!} x^{k-i}.$$

Therefore the **abstract Taylor expansion** of $y \mapsto y^k$ around x is

$$U(x) := \sum_{i=0}^{k} \binom{k}{i} x^{k-i} X^i = (X + x)^k \in \mathbb{R}[X].$$

Moreover we recover the function $y \mapsto y^k$

$$y^k = [\Pi_x U(x)]\,(y).$$

Change of the base point

If we set, for $x, z \in \mathbb{R}^d$, $\Gamma_{xz} : \mathbb{R}[X] \mapsto \mathbb{R}[X]$

$$\Gamma_{xz} X^k = (X + x - z)^k = \sum_{i=0}^{k} \binom{k}{i} (x - z)^{k-i} X^i,$$

then it is easy to see that

$$U(x) = \Gamma_{xz} U(z).$$

Indeed

$$U(x) = (X + x)^k = (X + z + x - z)^k = \Gamma_{xz} U(z).$$

Change of the base point

The operator

$$\Gamma_{xz} X^k = (X + x - z)^k = \sum_{i=0}^{k} \binom{k}{i} (x - z)^{k-i} X^i.$$

gives a rule to transform a classical Taylor expansion centered at z of a fixed polynomial into one centered at x.

This definition satisfies the simple properties

$$\Pi_z = \Pi_x \Gamma_{xz}, \qquad \Gamma_{xx} = \mathrm{Id}, \qquad \Gamma_{xy}\Gamma_{yz} = \Gamma_{xz},$$

$$\deg(\Gamma_{xz} X^k - X^k) < k \qquad \|\Gamma_{xz} X^k - X^k\|_i \leq C \|x - z\|^{k-i}.$$

Classical polynomials

Given a global function $y \mapsto y^k$, we can associate to each x its Taylor expansion around x

$$U(x) = (X + x)^k = \Gamma_{x0} X^k = \Gamma_{x0} U(0).$$

By linearity, we obtain that $U \mapsto \mathbb{R}[X]$ is the Taylor expansion of a (classical) polynomial $P(\cdot)$ if and only if

$$U(x) - \Gamma_{xz} U(z) \equiv 0$$

and in this case

$$U(z) = \sum_{i=0}^{deg(P)} \frac{P^{(i)}(z)}{i!} X^i.$$

In particular for all x, y, z

$$\Pi_x U(x)(y) \equiv \Pi_z U(z)(y) = P(y).$$

Hölder functions

A function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be of class $C^{k+\beta}$ if it is everywhere k-times differentiable with (bounded) derivatives and the k-th derivative is β-Hölder continuous.

In fact this is equivalent to requiring that for all x there exists a polynomial $P_x(\cdot)$ of degree k such that

$$|u(y) - P_x(y)| \leq C|y-x|^{k+\beta} \tag{1}$$

and in this case necessarily

$$P_x(y) = \sum_{i=0}^{k} \frac{u^{(i)}(x)}{i!}(y-x)^i = \Pi_x \left[\sum_{i=0}^{k} \frac{u^{(i)}(x)}{i!} X^i \right](y).$$

Hölder functions

If we define

$$U(x) = \sum_{i=0}^{k} \frac{u^{(i)}(x)}{i!} X^i \in \mathbb{R}[X],$$

then we obtain

$$U(x) - \Gamma_{xz} U(z) = \sum_{i=0}^{k} \frac{X^i}{i!} \left(u^{(i)}(x) - \sum_{j=0}^{k-i} \frac{u^{(i+j)}(z)}{j!} (x-z)^j \right)$$

and in particular $u \in C^{k+\beta}$ iff for all $i \leq k$

$$\|U(x) - \Gamma_{xz} U(z)\|_i \leq C\|x-z\|^{k+\beta-i}.$$

We say that $U \in \mathcal{D}^\gamma$ if $U : \mathbb{R}^d \to \mathbb{R}[X]$ takes values in the span of monomials with degree strictly less than γ and for all $i < \gamma$

$$\|U(x) - \Gamma_{xz} U(z)\|_i \leq C\|x-z\|^{\gamma-i}.$$

Differential equations

This gives a characterization of Hölder functions u in terms of their Taylor sum U and the operators Γ_{xz}. In general

$$u(x) = \Pi_x\, U(x)(x), \qquad (\textit{reconstruction})$$
$$u(y) - \Pi_x\, U(x)(y) \neq 0, \qquad \Pi_x\, U(x) \neq \Pi_z U(z).$$

For instance, if $d = 1$ then the ODE with α-Hölder coefficient $b : \mathbb{R} \to \mathbb{R}$

$$\frac{du}{dx} = b(u(x)), \qquad u(0) = u_0 \in \mathbb{R}$$

can be coded by $U \in \mathcal{D}^{1+\alpha}$ where

$$U(x) = u(x) + b(u(x))\, X, \qquad x \in \mathbb{R}.$$

Generalized Taylor expansions

Regularity Structures are a far-reaching generalization of the previous construction.

We want to add **new monomials** representing random distributions and to solve stochastic (partial) differential equations.

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For instance, let $\xi = \xi(x)$ is a space-time white noise on $\mathbb{R}^d$, i.e. a centered Gaussian field such that

$$\mathbb{E}(\xi(x)\xi(y)) = \delta(x-y), \qquad x, y \in \mathbb{R}^d.$$

A concrete realisation: for all $\psi \in L^2(\mathbb{R}^{d-1})$ and $t \in \mathbb{R}$

$$\int_{[0,t]\times\mathbb{R}^{d-1}} \psi(x)\,\xi(x)\,\mathrm{d}x := \sum_k B_k(t)\,\langle e_k, \psi\rangle,$$

where $(B_k)_k$ is an IID sequence of Brownian motions and $(e_k)_k$ is a complete orthonormal system in $L^2(\mathbb{R}^{d-1})$.

The stochastic heat equation

Let $v : \mathbb{R}^d \to \mathbb{R}$ solve the heat equation with external forcing

$$\partial_t v = \Delta v + \xi, \qquad x \in \mathbb{R}^d,$$

where

$$\partial_t = \partial_{x_1}, \qquad \Delta := \sum_{i=2}^{d} \partial_{x_i}^2.$$

The properties of this "process" depend heavily on the dimension, since

$$\mathrm{Var}(v(x)) = \int_0^t \frac{C_d}{s^{\frac{d-1}{2}}}\, ds \begin{cases} < +\infty, & d = 2 \\ = +\infty, & d \geq 3 \end{cases}$$

so that for $d \geq 3$ the solution is a random distribution.

Singular stochastic PDEs

If $\nabla = (\partial_{x_i}, i = 2, \ldots, d)$ then for a class of equations

$$\partial_t u = \Delta u + F(u, \nabla u, \xi), \qquad x \in \mathbb{R} \times \mathbb{R}^{d-1}$$

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$$\text{(KPZ)} \qquad \partial_t u = \Delta u + (\nabla u)^2 + \xi, \quad x \in \mathbb{R} \times \mathbb{R},$$

$$\text{(gKPZ)} \qquad \partial_t u = \Delta u + f(u)\,(\nabla u)^2 + g(u)\,\xi, \quad x \in \mathbb{R} \times \mathbb{R},$$

$$\text{(PAM)} \qquad \partial_t u = \Delta u + u\,\xi, \quad x \in \mathbb{R} \times \mathbb{R}^2,$$

$$(\Phi^4_3) \qquad \partial_t u = \Delta u - u^3 + \xi, \quad x \in \mathbb{R} \times \mathbb{R}^3.$$

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Even for polynomial non-linearities, we do not know how to properly define products of (random) distributions.

This is where infinities arise (see below).

Some notations: the heat kernel

Let $d \geq 2$.

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we define the heat kernel $G : \mathbb{R}^d \mapsto \mathbb{R}$

$$G(x) = \mathbb{1}_{(x_1>0)} \frac{1}{\sqrt{2\pi x_1}} \exp\left(-\frac{x_2^2 + \cdots + x_d^2}{2x_1}\right).$$

Given $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ we define

$$G^{(k)}(x) = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_d}}{\partial x_d^{k_d}} G(x).$$

Parabolic scaling

The heat kernel has a very important scaling property:

$$G(\delta^2 x_1, \delta x_2, \ldots, \delta x_d) = \frac{1}{\delta} G(x), \qquad \delta > 0.$$

This motivates the following definitions:

$$\|x - y\|_{\mathfrak{s}} := |x_1 - y_1|^{1/2} + |x_2 - y_2| + \cdots + |x_d - y_d|, \qquad x \in \mathbb{R}^d,$$

$$|k|_{\mathfrak{s}} := 2k_1 + k_2 + \cdots + k_d, \qquad k \in \mathbb{N}^2.$$

Generalized Monomials

We want to introduce new monomials which allow to approximate u locally.

We need a monomial for the noise : we introduce

$$\Xi, \qquad \Pi_x \Xi(y) := \xi(y).$$

Remember that $\Pi_x X^k(y) = (y-x)^k$ and

$$|\Pi_x X^k(y)| \leq \|x-y\|_{\mathfrak{s}}^{|k|_{\mathfrak{s}}}.$$

Then we see that the scaled degree $|k|_{\mathfrak{s}}$ of X^k has both an algebraic and an analytic interpretation.

We need a similar concept for all (abstract) monomials.

Abstract Monomials

We define the following family $\mathcal{T}$ of symbols (trees):

- $1, X \in \{X_1, \ldots, X_d\}, \Xi \in \mathcal{T}$
- if $\tau_1, \ldots, \tau_n \in \mathcal{T}$ then $\tau_1 \cdots \tau_n \in \mathcal{T}$ (commutative and associative product)
- if $\tau \in \mathcal{T}$ then $\mathcal{I}(\tau) \in \mathcal{T}$ and $\mathcal{I}_k(\tau) \in \mathcal{T}$ (formal convolution with the heat kernel differentiated k times)

Examples: $\mathcal{I}(\Xi), X^n \Xi \mathcal{I}_k(\Xi), \mathcal{I}((\mathcal{I}_1(\Xi))^2)$

To a symbol τ we associate a real number $|\tau|$ called its homogeneity:
$|\Xi| = \alpha < -(d+1)/2$, $|X_1| = 2$, $|X_2| = 1$, $|1| = 0$

$$|\tau_1 \cdots \tau_n| = |\tau_1| + \cdots + |\tau_n|, \quad |\mathcal{I}_k(\tau)| = |\tau| + 2 - |k|_{\mathfrak{s}}.$$

Let $\mathcal{H}$ be the space of linear combinations of elements in $\mathcal{T}$.

$\alpha < 0$ is chosen so that ξ is a.s. a distribution of order at least α.

The Π operators

We fix a bounded smooth function ξ and define recursively functions of $y \in \mathbb{R}^d$

$$\Pi\, 1(y) = 1, \qquad \Pi\, X(y) = y, \qquad \Pi\, \Xi(y) = \xi(y),$$

$$\Pi(\tau_1 \cdots \tau_n)(y) = \prod_{i=1}^{n} \Pi\tau_i(y),$$

$$\Pi\, \mathcal{I}_k(\tau)(y) = (G^{(k)} * \Pi\tau)(y).$$

These are global functions which include $y \mapsto y^k$.

The Π_x operators

We define recursively for $\tau \in \mathcal{T}$ continuous generalized monomials $\Pi_x\tau$ around the base point x

$$\Pi_x 1(y) = 1, \qquad \Pi_x X(y) = (y-x), \qquad \Pi_x \Xi(y) = \xi(y),$$

$$\Pi_x(\tau_1 \cdots \tau_n)(y) = \prod_{i=1}^{n} \Pi_x \tau_i(y),$$

$$\Pi_x \mathcal{I}_k(\tau)(y) = (G^{(k)} * \Pi_x \tau)(y) - \sum_{i=0}^{|\mathcal{I}_k(\tau)|} \frac{(y-x)^i}{i!} (G^{(i+k)} * \Pi_x \tau)(x).$$

Then $|\tau|$ is the analytical homogeneity of the monomial $\Pi_x\tau$:

$$|\Pi_x \tau(y)| \leq C \|y - x\|_{\mathfrak{s}}^{|\tau|}.$$

Beware: if ξ is white noise then products are (very) problematic and will have to be renormalized.

Regularity structures

Let us give an (almost) complete definition of a regularity structure $\mathcal{T}$ [Hairer '14]: this is a triplet $(A, \mathcal{H}, G)$ where

- $A \subset \mathbb{R}$ is an index set which contains 0 and which is locally finite and bounded below (the set of possible homogeneities)
- $\mathcal{H} = \oplus_{\alpha \in A} \mathcal{H}_\alpha$ is a graded vector space
- G, the Structure group, acts on $\mathcal{H}$ in such a way that for all $\Gamma \in G$, $\alpha \in A$ and $a \in \mathcal{H}_\alpha$

$$\Gamma a - a \in \bigoplus_{\beta < \alpha} \mathcal{H}_\beta.$$

G is one of the two main groups in the theory; its algebraic structure will be discussed in detail by Yvain.

General models

A model of $\mathcal{T}$ is given by a couple (Π_x, Γ_{xz}) such that

1. for all x, $\Pi_x : \mathcal{T} \mapsto \mathcal{S}'(\mathbb{R}^d)$ and for all $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$|\Pi_x \tau(\varphi_{x,\delta})| \leq C\delta^{|\tau|},$$

where $\varphi_{x,\delta}(z) := \frac{1}{\delta^{d+1}} \varphi\left(\delta^{-2}(z_1 - x_1), \delta^{-1}(z_i - x_i), i \geq 2\right).$

2. $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \to G$ is such that for all x, y, z

$$\Gamma_{xx} = \mathrm{Id}, \qquad \Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}, \qquad |\Gamma_{xz}\tau - \tau| < |\tau|$$

$$\|\Gamma_{xz}\tau - \tau\|_\ell \leq C\|z - x\|^{|\tau|-\ell}, \ \ell < |\tau|.$$

3. for all x, z: $\Pi_z = \Pi_x \Gamma_{xz}$.

Functional norm

In the general case, for $\gamma > 0$ we say that $U \in \mathcal{D}^\gamma$ if U takes values in the linear span of the symbols with homogeneity $< \gamma$ and for all $\beta < \gamma$

$$\|U(x) - \Gamma_{xy}U(y)\|_\beta \leq C_U \|x - y\|_{\mathfrak{s}}^{\gamma-\beta}$$

This is a notion of Hölder regularity with respect to generalized monomials.

If U takes values in sums of X^k, then the definition is equivalent to the classical C^γ-regularity (for $\gamma \notin \mathbb{N}$).

This definition is inspired by Massimiliano Gubinelli's theory of controlled rough paths.

We want to solve our SPDEs with some abstract fixed point in one of these Banach spaces.

The reconstruction theorem

Our starting problem was to associate to a function u a Taylor expansion $U(x)$ around each point x.

What about the inverse problem? Given such $x \mapsto U(x) \in \mathcal{H}$, can we find a function u with this expansion up to a remainder?

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This is the content of the Reconstruction Theorem:

For all $\gamma > 0$ *there exists a unique operator* $\mathcal{R} : \mathcal{D}^\gamma \mapsto \mathcal{S}'(\mathbb{R}^d)$ *s.t.*

$$|\mathcal{R}U(y) - \Pi_x U(x)(y)| \leq C_U \|x-y\|_{\mathfrak{s}}^\gamma \tag{2}$$

for all x, y*, or, more precisely, such that for* $\delta > 0$

$$|\mathcal{R}U(\varphi_{x,\delta}) - \Pi_x U(x)(\varphi_{x,\delta})| \leq C_U \delta^\gamma.$$

Note that (2) is the exact analog of (1): a Taylor expansion of $u := \mathcal{R}U$.

Regularisation of SPDEs

Let $\xi_\varepsilon = \rho_\varepsilon * \xi$ a regularisation of ξ and let u_ε solve

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \nabla u_\varepsilon, \xi_\varepsilon), \qquad x \in \mathbb{R}^d.$$

What happens as $\varepsilon \to 0$?

If we fix a Banach space of generalised functions $\mathcal{H}^{-\alpha}$ on $\mathbb{R}^d$ such that $\xi \in \mathcal{H}^{-\alpha}$ a.s. for some fixed $\alpha > 0$, then the map $\xi_\varepsilon \mapsto u_\varepsilon$ is not continuous.

We need a topology such that

- the map $\xi_\varepsilon \mapsto u_\varepsilon$ is continuous
- $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$.

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We need a topology such that

- the map $\xi_\varepsilon \mapsto u_\varepsilon$ is continuous
- $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$.

It turns out that the correct topology is, roughly speaking, the convergence of $(\Pi^\varepsilon_x, \Gamma^\varepsilon_{xz})$: this is a purely analytic statement.

The probabilistic statement is: "this works for ξ the white noise".

Convergence

Let us try the monomial $\Xi\mathcal{I}(\Xi)$. Then (for simplicity: Π instead of Π_x)

$$T_\varepsilon := \Pi^\varepsilon \Xi\mathcal{I}(\Xi)(\varphi) = \int \varphi(y)\, \xi_\varepsilon(y)\, (G * \xi_\varepsilon)(y)\, \mathrm{d}y$$

with $\varphi \in C_c^\infty(\mathbb{R}^d)$. Now

$$\mathbb{E}[T_\varepsilon] = \int \varphi(y)\, \mathbb{E}[\xi_\varepsilon (G * \xi_\varepsilon)](y)\, \mathrm{d}y = \int \varphi(y)\, \rho_\varepsilon * G * \rho_\varepsilon(0)\, \mathrm{d}y$$

and

$$\lim_{\varepsilon\to 0} \mathrm{Var}[T_\varepsilon] = \int \varphi^2(y)\, G^2(y-x)\, \mathrm{d}y\, \mathrm{d}x < +\infty.$$

However $\rho_\varepsilon * G * \rho_\varepsilon(0) \to +\infty$ as $\varepsilon \to 0$: a first example of the famous infinities which need renormalization. In this case

$$\xi_\varepsilon\, G * \xi_\varepsilon - \mathbb{E}[\xi_\varepsilon\, G * \xi_\varepsilon] = \xi_\varepsilon\, G * \xi_\varepsilon - \rho_\varepsilon * G * \rho_\varepsilon(0).$$

Products of (random) distributions

Diverging terms include

$$\xi_\varepsilon (G * \xi_\varepsilon), \quad (\partial_x G * \xi_\varepsilon)^2, \quad \xi_\varepsilon\, G * (\xi_\varepsilon\, G * \xi_\varepsilon), \quad \dots$$

They all tend to products of (random) distributions.

Indeed, the problems come from the (canonical) choice of imposing multiplicativity of the Π_x^ε operator in (19):

$$\Pi_x^\varepsilon(\tau_1 \cdots \tau_n)(y) = \prod_{i=1}^n \Pi_x^\varepsilon \tau_i(y).$$

This formula needs to be modified:

$$\hat{\Pi}_x^\varepsilon(\tau_1 \cdots \tau_n)(y) = \prod_{i=1}^n \hat{\Pi}_x^\varepsilon \tau_i(y) \; + \; ?$$

(we'll discuss later more precisely the ?).

Renormalization of the model

It is necessary to modify $(\Pi_x^\varepsilon, \Gamma_{xz}^\varepsilon)$. But how?

A simple *Ansatz* is to consider suitable linear operators $M_\varepsilon : \mathcal{H} \to \mathcal{H}$ and to look for $(\Pi_x^{M_\varepsilon}, \Gamma_{xz}^{M_\varepsilon})$ such that

$$\Pi^{M_\varepsilon}\tau = \Pi^\varepsilon M_\varepsilon \tau$$

(note: Π not Π_x) in such a way that $(\Pi_x^{M_\varepsilon}, \Gamma_{xz}^{M_\varepsilon})$ converges as $\varepsilon \to 0$.

Remember: must satisfy $\Pi_z = \Pi_x \Gamma_{xz}$.

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Remember: must satisfy $\Pi_z = \Pi_x \Gamma_{xz}$.

Theorem

There exists a finite-dimensional Lie group $\mathfrak{R}$ acting on $\mathcal{H}$ and deterministic $M_\varepsilon \in \mathfrak{R}$ such that the only model $(\Pi_x^{M_\varepsilon}, \Gamma_{xz}^{M_\varepsilon})$ satisfying

$$\Pi^{M_\varepsilon}\tau = \Pi^\varepsilon M_\varepsilon \tau$$

converges as $\varepsilon \to 0$.

Regularisation

Let $\xi_\varepsilon = \rho_\varepsilon * \xi$ a regularisation of the white noise ξ and let u_ε solve

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \nabla u_\varepsilon, \xi_\varepsilon), \qquad x \in \mathbb{R} \times \mathbb{R}^{d-1}.$$

What happens as $\varepsilon \to 0$?

- We introduce a model $(\Pi^\varepsilon_x, \Gamma^\varepsilon_{xz})$ as in (19)
- we associate to u_ε a Taylor expansion U_ε
- we show that U_ε solves a fixed point problem in some $\mathcal{D}^\gamma(\varepsilon)$
- we hope that everything converges as $\varepsilon \to 0$.

Technical remark: we can restrict all models to $\oplus_{\beta<\gamma}\mathcal{H}_\beta$, thus to a finite number of generalized monomials.

One of the main results of the Regularity Structures theory is that

- u is a continuous functional of (Π_x, Γ_{xz}) (see below).

However, does $(\Pi^\varepsilon_x, \Gamma^\varepsilon_{xz})$ converge as $\varepsilon \to 0$?

The solution map

The analytic part of the theory constructs a solution map

$$\Phi : \mathcal{M} \to \mathcal{S}'(\mathbb{R}^d)$$

where $\mathcal{M}$ is the space of possible (Π_x, Γ_{xz})'s of $\mathcal{T}$, such that

- Φ is continuous
- if $\xi \in C^\infty(\mathbb{R}^d)$ and $u = \Phi(\Pi_x, \Gamma_{xz})$, see (19), then

$$\partial_t u = \Delta u + F(u, \nabla u, \xi).$$

- in particular if $u_\varepsilon = \Phi(\Pi_x^\varepsilon, \Gamma_{xz}^\varepsilon)$ with $\xi_\varepsilon := \rho_\varepsilon * \xi$ then

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \nabla u_\varepsilon, \xi_\varepsilon).$$

Now, if $\hat{u}_\varepsilon := \Phi(\Pi_x^{M_\varepsilon}, \Gamma_{xz}^{M_\varepsilon})$, does $\hat{u}_\varepsilon$ satisfy an equation?

The renormalized equation

Amazingly, $\hat{u}_\varepsilon$ satisfies

$$\partial_t \hat{u}_\varepsilon = \Delta \hat{u}_\varepsilon + F_\varepsilon(\hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon, \xi_\varepsilon)$$

where F_ε is an explicit, deterministic modification of F.

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$$\partial_t \hat{u}_\varepsilon = \Delta \hat{u}_\varepsilon + F_\varepsilon(\hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon, \xi_\varepsilon)$$

where F_ε is an explicit, deterministic modification of F. Examples:

(KPZ) $$\partial_t \hat{u}_\varepsilon = \Delta \hat{u}_\varepsilon + (\nabla \hat{u}_\varepsilon)^2 - C_\varepsilon + \xi_\varepsilon, \quad x \in \mathbb{R} \times \mathbb{R},$$

(gKPZ) $$\partial_t \hat{u}_\varepsilon = \Delta \hat{u}_\varepsilon + f(\hat{u}_\varepsilon)\left((\nabla \hat{u}_\varepsilon)^2 - C_\varepsilon\right) + h_\varepsilon(\hat{u}_\varepsilon) + g(\hat{u}_\varepsilon)\left(\xi_\varepsilon - C_\varepsilon\, g'(\hat{u}_\varepsilon)\right), \quad x \in \mathbb{R} \times \mathbb{R},$$

(PAM) $$\partial_t \hat{u}_\varepsilon = \Delta \hat{u}_\varepsilon + \hat{u}_\varepsilon\, \xi_\varepsilon - C_\varepsilon, \quad x \in \mathbb{R} \times \mathbb{R}^2,$$

(Φ^4_3) $$\partial_t \hat{u}_\varepsilon = \Delta \hat{u}_\varepsilon - \hat{u}_\varepsilon^3 + (C^1_\varepsilon + C^2_\varepsilon)\, \hat{u}_\varepsilon + \xi_\varepsilon, \quad x \in \mathbb{R} \times \mathbb{R}^3.$$

The renormalized solution

The renormalization group $\mathfrak{R}$ acts on the possible limits $(\hat{\Pi}_x, \hat{\Gamma}_{xz})$ and therefore on the possible renormalized solutions $\hat{u} := \Phi(\hat{\Pi}_x, \hat{\Gamma}_{xz})$.

Therefore the renormalized solution is neither unique nor canonical. One can define for instance

$$\xi_\varepsilon(G * \xi_\varepsilon) \mapsto \xi_\varepsilon(G * \xi_\varepsilon) - \mathbb{E}[\xi_\varepsilon(G * \xi_\varepsilon)] + c$$

for any constant $c \in \mathbb{R}$ and this still defines a good limit.

Questions:

- does $\hat{u}$ satisfy an equation ?

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The renormalization group $\mathfrak{R}$ acts on the possible limits $(\hat{\Pi}_x, \hat{\Gamma}_{xz})$ and therefore on the possible renormalized solutions $\hat{u} := \Phi(\hat{\Pi}_x, \hat{\Gamma}_{xz})$.

Therefore the renormalized solution is neither unique nor canonical. One can define for instance

$$\xi_\varepsilon(G * \xi_\varepsilon) \mapsto \xi_\varepsilon(G * \xi_\varepsilon) - \mathbb{E}[\xi_\varepsilon(G * \xi_\varepsilon)] + c$$

for any constant $c \in \mathbb{R}$ and this still defines a good limit.

Questions:

- does $\hat{u}$ satisfy an equation ?

Answer:

- yes and no...

$\hat{U}$ satisfy an equation in $\mathcal{D}^\gamma$, $\hat{u}$ satisfies an equation with renormalized products.

Factorisation

The study of our singular SPDE

$$\partial_t u = \Delta u + F(u, \nabla u, \xi)$$

factorises into three different problems:

- (Analytic step) Construction and continuity of the solution map $\Phi : \mathcal{M} \to \mathcal{S}'(\mathbb{R}^d)$, where $\mathcal{M}$ is the space of models.
- (Algebraic step) Construction of the renormalization group $\mathfrak{R}$.
- (Probabilistic step) Convergence of the modified model $(\Pi_x^{M_\varepsilon}, \Gamma_{xz}^{M_\varepsilon})$ as $\varepsilon \to 0$ to an $\mathcal{M}$-valued random variable $(\hat{\Pi}_x, \hat{\Gamma}_{xz})$ that we call the renormalized model.

Graph representation

Recall that, by the definition (18), the Π^ε's are polynomial functions of ξ_ε.

We have now N random variables $P_1(\xi_\varepsilon), \ldots, P_N(\xi_\varepsilon)$, polynomial functions of ξ_ε.

More precisely, for a fixed $\varphi \in C_c^\infty$ we consider the random variables

$$Z_i := \int_{\mathbb{R}^d} \varphi(z)\, P_i(\xi_\varepsilon(z))\, dz, \qquad i = 1, \ldots, N.$$

To each such random variable we associate a rooted tree T_i.

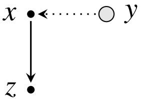
Every integration variable in Z_i is a vertex in T_i.

Every integral kernel in Z_i is an edge in T_i.

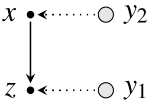
Examples

$$\Xi \longrightarrow \int \varphi(z)\, \xi_\varepsilon(z)\, dz = \int \varphi(z)\, \rho_\varepsilon(z-x)\, \xi(dx)\, dz \quad \longrightarrow$$

$$\mathcal{I}(\Xi) \longrightarrow \int \varphi(z)\, G * \xi_\varepsilon(z)\, dz \quad \longrightarrow$$

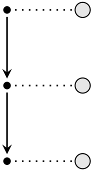


$$\Xi\mathcal{I}(\Xi) \longrightarrow \int \varphi(z)\, \xi_\varepsilon(z)\, G * \xi_\varepsilon(z)\, dz \quad \longrightarrow$$

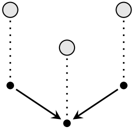


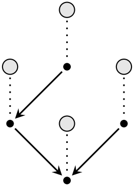
Examples

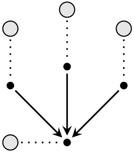
$\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))$

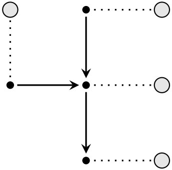


$\Xi\mathcal{I}(\Xi)^2$







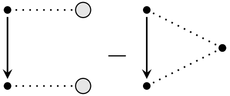


Feynman diagrams

Do you remember? We noticed that $\xi_\varepsilon \, G * \xi_\varepsilon$ can be renormalised by subtracting its expectation:

$$\xi_\varepsilon \, G * \xi_\varepsilon - \mathbb{E}[\xi_\varepsilon \, G * \xi_\varepsilon] = \xi_\varepsilon \, G * \xi_\varepsilon - \rho_\varepsilon * G * \rho_\varepsilon(0).$$

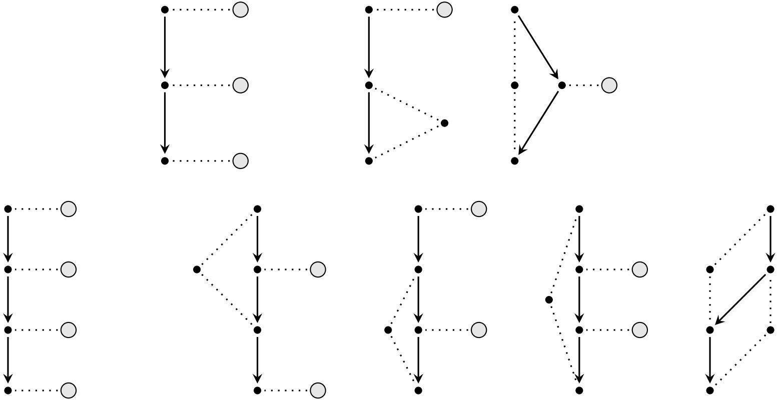
In terms of graphs (Feynman diagrams), this can be written as



Note that graphically the second graph is obtained from the first after a contraction of two leaves.

Feynman diagrams

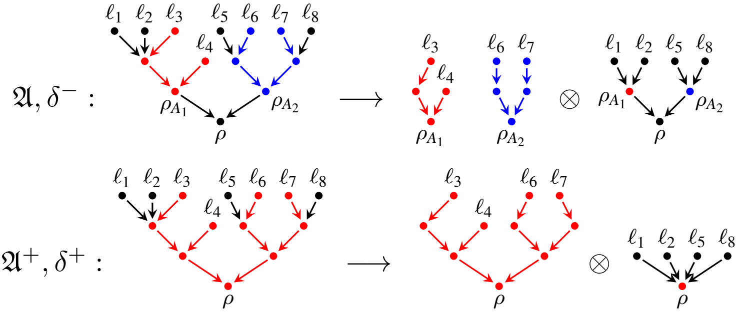
Other contractions:



Computation with $\bar{\Delta}$

Let $\bar{\mathfrak{A}} \subset \mathfrak{A}$, we define infinite triangular linear maps

$$\bar{\Delta} F^{\mathfrak{n}}_{\mathfrak{e}} = \sum_{\mathcal{A} \in \bar{\mathfrak{A}}(F)} \sum_{\mathfrak{n}_{\mathcal{A}}, \mathfrak{e}_{\mathcal{A}}} \frac{1}{\mathfrak{e}_{\mathcal{A}}!} \binom{\mathfrak{n}}{\mathfrak{n}_{\mathcal{A}}} \mathcal{R}^{\uparrow}_{\mathcal{A}} F^{\mathfrak{n}_{\mathcal{A}} + \pi \mathfrak{e}_{\mathcal{A}}}_{\mathfrak{e}} \otimes \mathcal{R}^{\downarrow}_{\mathcal{A}} F^{\mathfrak{n} - \mathfrak{n}_{\mathcal{A}}}_{\mathfrak{e} + \mathfrak{e}_{\mathcal{A}}}$$



Renormalisation groups

Recall that

$$\delta^+ T^{\mathfrak{n}}_{\mathfrak{e}} = \sum_{\mathcal{A}\in\mathfrak{A}^+(T)} \sum_{\mathfrak{n}_\mathcal{A},\mathfrak{e}_\mathcal{A}} \frac{1}{\mathfrak{e}_\mathcal{A}!} \binom{\mathfrak{n}}{\mathfrak{n}_\mathcal{A}} \mathcal{R}^{\uparrow}_{\mathcal{A}} T^{\mathfrak{n}_\mathcal{A}+\pi\mathfrak{e}_\mathcal{A}}_{\mathfrak{e}} \otimes \mathcal{R}^{\downarrow}_{\mathcal{A}} T^{\mathfrak{n}-\mathfrak{n}_\mathcal{A}}_{\mathfrak{e}+\mathfrak{e}_\mathcal{A}}$$

$$\Delta = (\mathrm{id} \otimes \Pi_+)\delta^+, \qquad \Delta^+ = (\Pi_+ \otimes \Pi_+)\delta^+.$$

Now

$$\delta^- T^{\mathfrak{n}}_{\mathfrak{e}} = \sum_{\mathcal{A}\in\mathfrak{A}^-(T)} \sum_{\mathfrak{n}_\mathcal{A},\mathfrak{e}_\mathcal{A}} \frac{1}{\mathfrak{e}_\mathcal{A}!} \binom{\mathfrak{n}}{\mathfrak{n}_\mathcal{A}} \mathcal{R}^{\uparrow}_{\mathcal{A}} T^{\mathfrak{n}_\mathcal{A}+\pi\mathfrak{e}_\mathcal{A}}_{\mathfrak{e}} \otimes \mathcal{R}^{\downarrow}_{\mathcal{A}} T^{\mathfrak{n}-\mathfrak{n}_\mathcal{A}}_{\mathfrak{e}+\mathfrak{e}_\mathcal{A}}$$

$$\hat{\Delta} = (\Pi_- \otimes \mathrm{id})\delta^-, \qquad \Delta^- = (\Pi_- \otimes \Pi_-)\delta^-.$$

Renormalisation groups

Positive renormalization:
$G := \{g \in \mathcal{H}_+^* : g(\tau_1\tau_2) = g(\tau_1)g(\tau_2), \quad \forall\, \tau_1, \tau_2 \in \mathcal{H}_+\}$,

$$\Gamma_g : \mathcal{H} \to \mathcal{H}, \qquad \Gamma_g\tau := (\mathrm{id} \otimes g)\Delta\tau$$
$$\Gamma_g\Gamma_{\hat{g}} = \Gamma_{g'}, \qquad \Gamma_{g'}\tau := (g \otimes \hat{g})\Delta^+\tau$$

Negative renormalization:
$\mathfrak{R} := \{\ell \in \mathcal{H}_-^* : g(\tau_1\tau_2) = g(\tau_1)g(\tau_2), \quad \forall\, \tau_1, \tau_2 \in \mathcal{H}_-\}$

$$M_\ell : \mathcal{H} \to \mathcal{H}, \qquad M_\ell\tau := (\ell \otimes \mathrm{id})\hat{\Delta}\tau$$
$$M_\ell M_{\hat{\ell}} = M_{\ell'}, \qquad M_{\ell'}\tau := (\ell \otimes \hat{\ell})\Delta^-\tau$$

Note that G and $\mathfrak{R}$ depend on the equation.

Nilpotency of Renormalisation groups

Note that

- for all $\Gamma \in G$ and $\tau \in \mathcal{H}_\alpha$,

$$\Gamma\tau - \tau \in \bigoplus_{\beta<\alpha} \mathcal{H}_\beta.$$

- for all $M \in \mathfrak{R}$ and $\tau \in \mathcal{H}_\alpha$,

$$M\tau - \tau \in \bigoplus_{\beta>\alpha} \mathcal{H}_\beta.$$

The last property is the reason why in general $\Pi_x^M \neq \Pi_x M$.

Hopf algebras

We have presented several algebraic constructions based on extraction/contraction of labelled forests.

This works well but only up to a certain point. In fact this operation entails a certain loss of information. There are several possible definitions of different regularity structures which retain the necessary information.

Instead of extracting/contracting, we can choose a different operation: if F is a finite set, then we can consider the set of pairs (B, A) with $A \subseteq B \subseteq F$ and

$$\Delta(B, A) := \sum_{A \subseteq C \subseteq B} (C, A) \otimes (B, C).$$

Then it is easy to see that this operation is co-associative

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta.$$

Labelled forests

Now we suppose that F is a forest and $A, \hat{F}$ are subforests with $\hat{F} \subseteq A \subseteq F$. Then

$$\Delta(F, \hat{F}) := \sum_{\hat{F} \subseteq A \subseteq F} (A, \hat{F}) \otimes (F, A)$$

is similar to the operation of extraction/contraction but without loss of information.

How can we add labels? Recall that

- nodes represent integration variables
- edges represent integration kernels
- node-labels represent powers of the integration variables
- edge-labels represent derivatives of the integration kernels.

Labelled forests

One possible choice is to work on the space $\mathfrak{F} := \{(F, \hat{F}, \mathfrak{n}, \hat{\mathfrak{n}}, \mathfrak{e})\}$ where

1. $\hat{F}$ is a subforest of F
2. $\mathfrak{n}$ is an $\mathbb{N}^d$-valued function on the node set N_F of F
3. $\hat{\mathfrak{n}}$ is a $\mathbb{Z}^d$-valued function on N_F with support in the node set $N_{\hat{F}}$ of $\hat{F}$
4. $\mathfrak{e}$ is an $\mathbb{N}^d$-valued function on the edge set E_F of F with support in $E_F \setminus E_{\hat{F}}$.

For $\varepsilon : E_F \to \mathbb{N}^d$ we define $\pi\varepsilon : N_F \to \mathbb{N}^d$

$$\pi\varepsilon(x) := \sum_{e=(x,y)\in E_F} \varepsilon(e).$$

Coproduct

$$\bar{\Delta}(F,\hat{F},\mathfrak{n},\hat{\mathfrak{n}},\mathfrak{e})$$
$$:= \sum_{A\in\bar{\mathfrak{A}}(F,\hat{F})}\sum_{\varepsilon_A,\mathfrak{n}_A}\frac{1}{\varepsilon_A!}\binom{\mathfrak{n}}{\mathfrak{n}_A}(A,\hat{F},\mathfrak{n}_A+\pi\varepsilon_A,\hat{\mathfrak{n}},\mathfrak{e})\otimes$$
$$\otimes\,(F,A,\mathfrak{n}-\mathfrak{n}_A,\ \hat{\mathfrak{n}}+\mathfrak{n}_A+\pi(\varepsilon_A-\mathfrak{e}_\emptyset^A),\mathfrak{e}_A+\varepsilon_A)\ ,$$

where

- $\bar{\mathfrak{A}}(F,\hat{F})$ is a class of subforests of F containing $\hat{F}$
- for a subforest A of F we denote $\mathfrak{e}_A := \mathfrak{e}\restriction_{E_F\setminus E_A}$
- $\mathfrak{n}_A$ runs over all $\mathfrak{n}_A : N_F \to \mathbb{N}^d$ supported by N_A
- ε_A runs over all $\varepsilon_A : E_F \to \mathbb{N}^d$ supported on the set of edges

$$\partial(F,A) := \{(e_+,e_-)\in E_F\setminus E_A\ :\ e_+\in N_A\}\,.$$

Note that $\bar{\Delta}$ is defined by an infinite sum, since ε_A is unconstrained.

Coproduct

The construction on couples of forests:

$$\bar{\Delta}(F, \hat{F}, \mathfrak{n}, \hat{\mathfrak{n}}, \mathfrak{e})$$
$$:= \sum_{A \in \bar{\mathfrak{A}}(F,\hat{F})} \sum_{\varepsilon_A, \mathfrak{n}_A} \frac{1}{\varepsilon_A!} \binom{\mathfrak{n}}{\mathfrak{n}_A} (A, \hat{F}, \mathfrak{n}_A + \pi\varepsilon_A, \hat{\mathfrak{n}}, \mathfrak{e}) \otimes$$
$$\otimes \left(F, A, \mathfrak{n} - \mathfrak{n}_A,\ \hat{\mathfrak{n}} + \mathfrak{n}_A + \pi(\varepsilon_A - \mathfrak{e}_\emptyset^A), \mathfrak{e}_A + \varepsilon_A\right) ,$$

the construction on forests is

$$\bar{\Delta} F_{\mathfrak{e}}^{\mathfrak{n}} = \sum_{A \in \bar{\mathfrak{A}}(F)} \sum_{\mathfrak{n}_A, \varepsilon_A} \frac{1}{\varepsilon_A!} \binom{\mathfrak{n}}{\mathfrak{n}_A} \mathcal{R}_A^{\uparrow} F_{\mathfrak{e}}^{\mathfrak{n}_A + \pi\varepsilon_A} \otimes \mathcal{R}_A^{\downarrow} F_{\mathfrak{e}+\varepsilon_A}^{\mathfrak{n}-\mathfrak{n}_A}$$

(see also the extended structure in Yvain's second lecture).

Coassociativity

Under some assumptions on $\bar{\mathfrak{A}}(F, \hat{F})$, we have

$$(\bar{\Delta} \otimes \mathrm{id})\bar{\Delta} = (\mathrm{id} \otimes \bar{\Delta})\bar{\Delta}.$$

This is in particular true in two special cases:

- $\mathfrak{A}^-(F, \hat{F}) := \{\text{all forests } A : \hat{F} \subseteq A \subseteq F\}$
- $\mathfrak{A}^+(F, \hat{F}) := \{$all forests $A : \hat{F} \subseteq A \subseteq F$, and for every connected component T of F, $T \cap A$ is a tree containing the root of $T\}$.

We call δ^- and δ^+ the corresponding operators.

Double coassociativity

There is a way to reformulate the previous construction so that

$$\mathcal{M}^{(13)(2)(4)}\big(\delta^- \otimes \delta^-\big)\delta^+ = (\mathrm{id} \otimes \delta^+)\delta^- \ ,$$

on $\mathfrak{F}$, where we used the notation

$$\mathcal{M}^{(13)(2)(4)}\big(\tau_1 \otimes \tau_2 \otimes \tau_3 \otimes \tau_4\big) = \big(\tau_1 \cdot \tau_3 \otimes \tau_2 \otimes \tau_4\big) \ .$$

This allows to define an explicit action of the renormalization group on the structure group of a regularity structure.

(See [D. Calaque, K. Ebrahimi-Fard and D. Manchon, 2011] for another appearance of this formula).

The advantage of this construction is its universality. For each equation, by a projection one finds the correct Hopf algebra/co-module.

Back to Taylor expansions

In the case of the positive renormalization, Yvain has already mentioned the following formula:

$$\Pi_x \tau = (\Pi \otimes f_x)\Delta\tau = \Pi\,\Gamma_{f_x}\tau$$

where f_x is suitably defined. Moreover $\Gamma_{xy} = \Gamma_{f_x}^{-1}\Gamma_{f_y}$.

This formula relates two canonical objects, Π and Π_x, via the positive renormalization.

Taylor expansions and negative renormalization

Let $T^{\mathfrak{n}}_{\mathfrak{e}}$ be a labelled tree. We recall that the renormalised $\hat{\Pi}^{\varepsilon}$ is given by

$$\hat{\Pi}^{\varepsilon} T^{\mathfrak{n}}_{\mathfrak{e}} = \Pi M_{\varepsilon} T^{\mathfrak{n}}_{\mathfrak{e}} =$$
$$= \sum_{\mathcal{A}\in\mathfrak{A}(T)} \sum_{\mathfrak{e}_{\mathcal{A}},\mathfrak{n}_{\mathcal{A}}} \frac{1}{\mathfrak{e}_{\mathcal{A}}!} \binom{\mathfrak{n}}{\mathfrak{n}_{\mathcal{A}}} \ell_{\varepsilon}\left(\Pi_{-}\mathcal{R}^{\uparrow}_{\mathcal{A}} T^{\mathfrak{n}_{\mathcal{A}}+\pi\mathfrak{e}_{\mathcal{A}}}_{\mathfrak{e}}\right) \Pi\mathcal{R}^{\downarrow}_{\mathcal{A}} T^{\mathfrak{n}-\mathfrak{n}_{\mathcal{A}}}_{\mathfrak{e}+\mathfrak{e}_{\mathcal{A}}}.$$

This is a (random) function on $(\mathbb{R}^d)^{N_T}$.

Let us suppose that T contains exactly n subtrees $T_i \subset T$ such that $r_i := -|(T_i)^0_{\mathfrak{e}}| > 0$ and that they are pairwise disjoint.

We set for $i = 1, \ldots, n$

$$F_i(y_v, v\in N_{T_i}) := \prod_{v\in N_{T_i}\setminus\{\rho_{T_i}\}} (y_v)^{\mathfrak{n}(v)} \prod_{e\in E_{\partial T_i}} G^{(\mathfrak{e}(e))}(y_{e_+} - y_{e_-}).$$

Taylor expansions and negative renormalization

Now for $F : \mathbb{R}^{dN} \to \mathbb{R}$, $r \in \mathbb{R}$, $v \in \mathbb{R}^{dN}$, we define $\mathfrak{T}_{r,v}K : \mathbb{R}^{dN} \to \mathbb{R}$ as

$$\mathfrak{T}_{r,v}F(y) := F(y) - \sum_{0 \le |j|_\mathfrak{s} < r} \frac{(y-v)^j}{j!} F^{(j)}(v),$$

namely $\mathfrak{T}_{r,v}F$ is the remainder of the Taylor expansion of F of order r around v. Then we find

$$\hat{\Pi}^\varepsilon T^\mathfrak{n}_\mathfrak{e}(y_v, v \in N_T) =$$

$$\prod_{v \notin \cup_i N_{T_i}} (y_v)^{\mathfrak{n}(v)} \prod_{e \in E_T \setminus \cup_i E_{\partial T_i}} G^{(\mathfrak{e}(e))}(y_{e_+} - y_{e_-}) \prod_{i=1}^n \mathfrak{T}_{r_i', y_{\rho_{T_i}}} F_i(y_v, v \in N_{T_i})$$

where for $i = 1, \ldots, n$

$$F_i(y_v, v \in N_{T_i}) := \prod_{v \in N_{T_i} \setminus \{\rho_{T_i}\}} (y_v)^{\mathfrak{n}(v)} \prod_{e \in E_{\partial T_i}} G^{(\mathfrak{e}(e))}(y_{e_+} - y_{e_-}).$$

The BPHZ formula

The previous result is called in QFT the BPHZ renormalization and is due to Bogoliubov-Parasiuk-Hepp-Zimmermann. (See Ajay Chandra's talk tomorrow)

The BPHZ formula

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That's fine for me: the only problem is the P.

The end

Thanks