Renormalizable Tensorial Field Theories as Models of Quantum Geometry

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"Paths to, from and in renormalization"

Purpose of this talk

Give you an impression of what are Tensorial Field Theories, and why people study them.

Figure : "Potsdamer Platz bei Nacht", Lesser Ury, 1920s

Group Field Theory: what is it?

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- The group manifold is **auxiliary**: should not be interpreted as space-time!
- Rather, **the Feynman amplitudes** are thought of as describing **space-time processes** $\rightarrow$ QFT *of* space-time rather than *on* space-time.
- Specific non-locality: determines the **combinatorial structure** of space-time processes (graphs, 2-complexes, triangulations...).

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Recommended reviews:

L. Freidel, "Group Field Theory: an overview", 2005
D. Oriti, "The microscopic dynamics of quantum space as a group field theory", 2011

General structure of a GFT and long-term objectives

Typical form of a GFT: field $\varphi(g_1, \dots, g_d)$, $g_\ell \in G$, with partition function

$$Z = \int [\mathcal{D}\varphi]_\Lambda \exp\left(-\varphi \cdot \mathcal{K} \cdot \varphi + \sum_{\{\mathcal{V}\}} t_{\mathcal{V}} \, \mathcal{V} \cdot \varphi^{n_{\mathcal{V}}} \right) = \sum_{k_{\mathcal{V}_1}, \dots, k_{\mathcal{V}_i}} \prod_i (t_{\mathcal{V}_i})^{k_{\mathcal{V}_i}} \{\text{SF amplitudes}\}$$

Main objectives of the GFT research programme:

1. Model building: define the **theory space**.
   e.g. spin foam models + combinatorial considerations (tensor models) $\rightarrow d$, G, $\mathcal{K}$ *and* $\{\mathcal{V}\}$.

2. Perturbative definition: prove that the spin foam expansion is **consistent** in some range of Λ.
   e.g. perturbative multi-scale renormalization.

3. Systematically explore the theory space: **effective continuum regime reproducing GR** in some limit?
   e.g. functional RG, constructive methods, condensate states...

- Partition function for $N \times N$ **symmetric matrix**:

$$\mathcal{Z}(N,\lambda) = \int [\mathrm{d}M] \exp\left(-\frac{1}{2}\mathrm{Tr} M^2 + \frac{\lambda}{N^{1/2}} \mathrm{Tr} M^3\right)$$

- **Large N expansion** $\rightarrow$ ensembles of **combinatorial maps**:

$$\mathcal{Z}(N,\lambda) = \sum_{\text{triangulation } \Delta} \frac{\lambda^{n_\Delta}}{s(\Delta)} \mathcal{A}_\Delta(N) = \sum_{g \in \mathbb{N}} N^{2-2g} \mathcal{Z}_g(\lambda)$$

- **Continuum limit** of $\mathcal{Z}_0$: tune $\lambda \rightarrow \lambda_c \Rightarrow$ very refined triangulations dominate.
$(\mathcal{Z}_0(\lambda) \sim |\lambda - \lambda_c|^{2-\gamma})$

Matrix models and random surfaces [t'Hooft, Polyakov, Kazakov, David,... '70s '80s]

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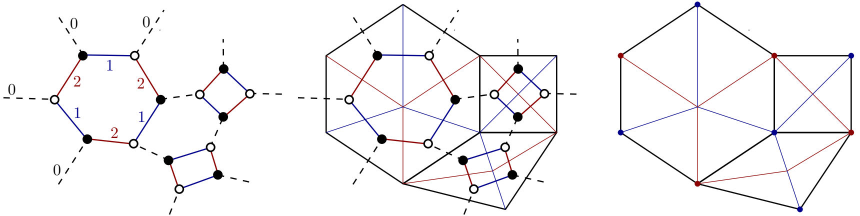
⇒ definition of **universal 2d random geometries**:

- do not depend on the details of the discretization, i.e. on the type of trace invariants used in the action;
- similarly, **Brownian map** rigorously constructed as a scaling limit of infinite triangulations and $2p$-angulations of the sphere. [Le Gall, Miermont '13]

Gluing of $2p$-angles:



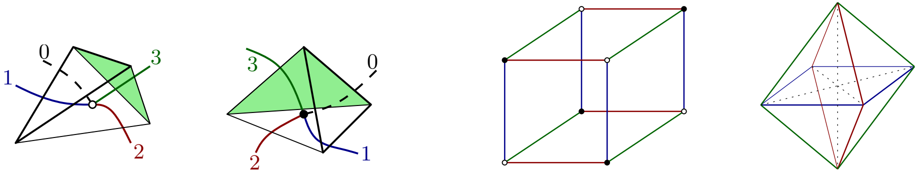
Duality:

3−colored graph	$\longleftrightarrow$	colored triangulation
node	$\longleftrightarrow$	triangle
line	$\longleftrightarrow$	edge
bicolored cycle	$\longleftrightarrow$	vertex

Any orientable surface with boundaries can be represented by such a **3-colored graph**.

Colored cell decompositions of pseudo-manifolds

Works in any dimension, e.g. in 3d:



Colored structure ⇒ unambiguous prescription for how to glue d-simplices along their sub-simplices.

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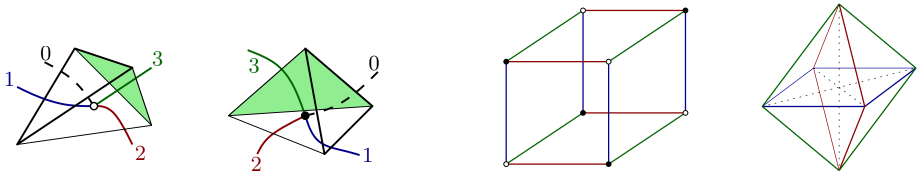

Colored structure $\Rightarrow$ unambiguous prescription for how to glue d-simplices along their sub-simplices.

$$\begin{aligned} (d+1)\text{-colored graph} &\quad\longleftrightarrow\quad \text{colored triangulation of dimension } d \\ \text{node} &\quad\longleftrightarrow\quad d\text{-simplex} \\ \text{connected component with } k \text{ colors} &\quad\longleftrightarrow\quad (d-k)\text{-simplex} \end{aligned}$$

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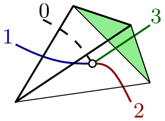
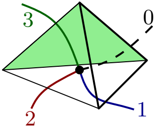
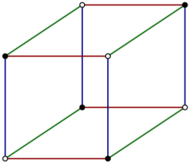
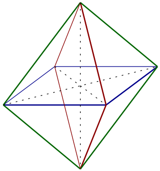








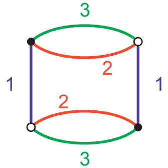
Colored structure ⇒ unambiguous prescription for how to glue d-simplices along their sub-simplices.

$(d+1)-$colored graph	$\longleftrightarrow$	colored triangulation of dimension d
node	$\longleftrightarrow$	$d-$simplex
connected component with k colors	$\longleftrightarrow$	$(d-k)-$simplex

Theorem: [Pezzana '74] Any PL manifold can be represented by a colored graph. In general, a $(d+1)$-colored graph represents a triangulated **pseudo-manifold** of dimension d.

⇒ **Crystallisation theory** [Cagliardi, Ferri et al. '80s]
Only recently introduced in GFTs / tensor models [Gurau '09...]

Trace invariants of fields $\varphi(g_1, g_2, \ldots, g_d)$ labelled by d-colored **bubbles** b:



$$\mathrm{Tr}_b(\varphi, \overline{\varphi}) = \int [\mathrm{d}g_i]^6 \, \overline{\varphi}(g_6, g_2, g_3)\varphi(g_1, g_2, g_3) \varphi(g_6, g_4, g_5)\overline{\varphi}(g_1, g_4, g_5)$$

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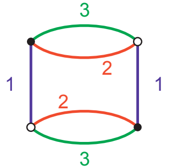

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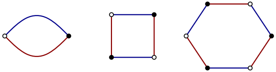

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Trace invariants

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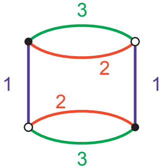

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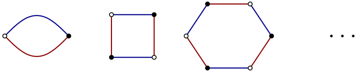

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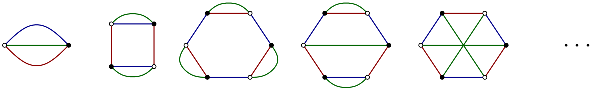

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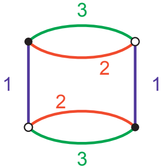

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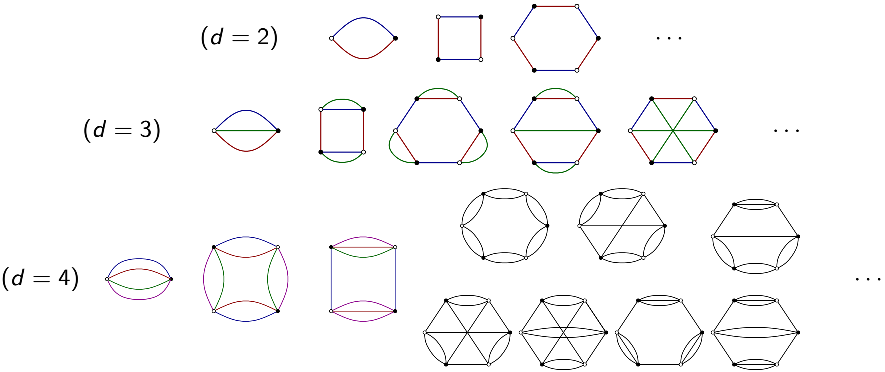

- **Tensor Models**: $T_{i_1 \cdots i_d}$, $i_k \in \{1, \ldots, N\}$
  - **$1/N$ expansion** dominated by spheres [Gurau '11...];
  - **continuum limit** of the leading order [Bonzom, Gurau, Riello, Rivasseau '11] $\rightarrow$ 'branched polymer' [Gurau, Ryan '13];
  - **double-scaling** limit [Dartois, Gurau, Rivasseau '13; Gurau, Schaeffer '13; Bonzom, Gurau, Ryan, Tanasa '14];
  - **Schwinger-Dyson** equations [Gurau '11 '12; Bonzom '12];
  - **non-perturbative** results [Gurau '11 '13; Delepouve, Gurau, Rivasseau '14];
  - **'multi-orientable'** models [Tanasa '11, Dartois, Rivasseau, Tanasa '13; Raasaakka, Tanasa '13; Fusy, Tanasa '14], **$O(N)^{\otimes d}$-invariant** models [SC, Tanasa '15], and **new scalings** [Bonzom '12; Bonzom, Delepouve, Rivasseau '15];
  - **symmetry breaking** to matrix phase [Benedetti, Gurau '15];
  - ...

- **Tensorial Group Field Theories**: $\varphi(g_1, \ldots, g_d)$, $g_\ell \in G$.
  - **Derivative operators** and non-trivial renormalization [Ben Geloun, Rivasseau '11...]
  - **Asymptotic freedom** [Ben Geloun '12...]
  - Heavier use of the group structure: **spin foam constraints** [Oriti, Rivasseau, SC '12 '13...]

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Mathematical objective: step-by-step generalization of standard renormalization techniques, until we are able to tackle 4d quantum gravity proposals.

Definition of TGFTs

- Ansatz akin to a **'local potential approximation'**:

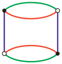
$$S_\Lambda(\varphi,\overline{\varphi}) = \overline{\varphi}\cdot\left(-\sum_\ell \Delta_\ell\right)\cdot\varphi + S^{\text{int}}_\Lambda(\varphi,\overline{\varphi})$$

- Subtlety: invariance properties on φ imposed by **spin foam constraints**.
- **Partition function**: (cut-off $\sum_{\ell=1}^{d} p_\ell^2 \lesssim \Lambda^2$)

$$\mathcal{Z}_\Lambda = \int \mathrm{d}\mu_{C_\Lambda}(\varphi,\overline{\varphi})\, \mathrm{e}^{-S^{\text{int}}_\Lambda(\varphi,\overline{\varphi})}\,.$$

  - $S^{\text{int}}_\Lambda(\varphi,\overline{\varphi})$ is **local**:

$$S^{\text{int}}_\Lambda(\varphi,\overline{\varphi}) = \sum_{b\in\mathcal{B}} t_b^\Lambda\, \mathrm{Tr}_b(\varphi,\overline{\varphi}) \underset{d=3}{=} t_2^\Lambda$$

$$+\, t_4^\Lambda$$

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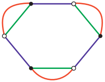

$$+\ \dots$$

  - Gaussian measure $\mathrm{d}\mu_C$ with possibly degenerate covariance:

$$C = \mathcal{P}\left(-\sum_\ell \Delta_\ell\right)^{-1}\mathcal{P}$$

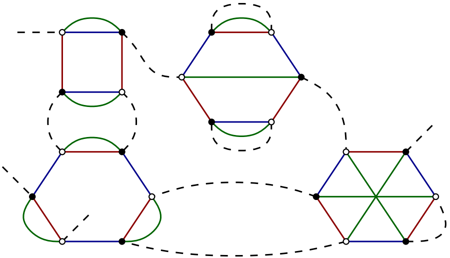
where $\mathcal{P}$ is a **projector** implementing the relevant constraints on the fields.

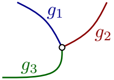
Feynman amplitudes

- Perturbative expansion in the coupling constants t_b:

$$\mathcal{Z} = \sum_{\mathcal{G}} \left(\prod_{b \in \mathcal{B}} (-t_b)^{n_b(\mathcal{G})} \right) \mathcal{A}_{\mathcal{G}}$$

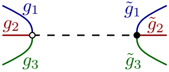
- Feynman graphs $\mathcal{G}$:





$= \int dg_1\, dg_2\, dg_3\, \ldots$

$= \delta(g\tilde{g}^{-1})$



$= C(g_1, g_2, g_3; \tilde{g}_1, \tilde{g}_2, \tilde{g}_3)$

- **Covariances** associated to the dashed, color-0 lines.
- **Face of color** ℓ = connected set of (alternating) color-0 and color-ℓ lines.

- **Gauge invariance condition**

$$\forall h \in G\,, \qquad \varphi(g_1, \ldots, g_d) = \varphi(g_1 h, \ldots, g_d h)$$

Common to **all** Spin Foam models: introduces a dynamical discrete connection at the level of the amplitudes.

TGFTs with gauge invariance condition

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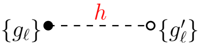
- Resulting propagator, including a regulator Λ ($\sim \sum_\ell p_\ell^2 \leq \Lambda^2$):

$$C_\Lambda(g_\ell; g'_\ell) = \int_{\Lambda^{-2}}^{+\infty} \mathrm{d}\alpha \int \mathrm{d}h \prod_{\ell=1}^{d} K_\alpha(g_\ell h g'^{-1}_\ell)\,,$$



where K_α is the **heat kernel** on G at time α.

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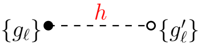

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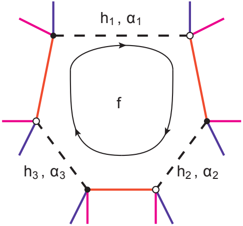
- The amplitudes are best expressed in terms of the **faces** of the Feynman graphs:



$$\longleftrightarrow \qquad K_{\alpha_1+\alpha_2+\alpha_3}(h_1 h_2 h_3)$$

Overview

<u>Goal:</u> check that the perturbative expansion - and henceforth **the connection to spin foam models** - is consistent.

- Types of models considered so far:
  - **'combinatorial'** models on $\mathrm{U}(1)^D$ $\rightarrow$ non-trivial propagators, but group structure otherwise auxiliary;
    [Ben Geloun, Rivasseau '11; Ben Geloun, Ousmane Samary '12; Ben Geloun, Livine '12...]
  - models with **'gauge invariance'** on $\mathrm{U}(1)^D$ and $\mathrm{SU}(2)$ $\rightarrow$ non-trivial propagators + one key dynamical ingredient of spin foam models.
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- Methods:
  - **multiscale analysis**: allows to rigorously prove renormalizability at **all orders in perturbation theory**;
  - **Connes–Kreimer** algebraic methods [Raasakka, Tanasa '13; Avohou, Rivasseau, Tanasa '15];
  - **loop-vertex expansion**: non-perturbative method allowing to resum the perturbative series [Gurau, Rivasseau,... '13].

Power-counting theorem

Goal: classify divergences according to the combinatorial properties of the graphs.

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Theorem

If G has dimension D, the UV divergences arise from subgraphs $\mathcal{H}$ with **degree of divergence**

$$\omega(\mathcal{H}) \geq 0\,,$$

where ω is defined by

- $\omega = -2L + D\,F$ in a model without gauge inv. condition; [Ben Geloun, Rivasseau '11]
- $\omega = -2L + D\,(F - R)$ in a model with gauge inv. condition; [Oriti, Rivasseau, SC '12]

and $R(\mathcal{H})$ is the rank of the **incidence matrix** between lines and faces of $\mathcal{H}$.

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Idea of proof: Multiscale analysis

- Decompose propagators:

$$C = \int \mathrm{d}\alpha \ldots = \sum_{i \in \mathbb{N}} \int_{M^{-2i}}^{M^{-2(i-1)}} \mathrm{d}\alpha \ldots = \sum_{i \in \mathbb{N}} C_i$$

- Decompose amplitudes according to $\mu = \{i_e\}$: $\mathcal{A}_{\mathcal{G}} = \sum_{\mu} \mathcal{A}_{\mathcal{G},\mu}$.
- Optimize single-slice bounds according to $\mu \rightarrow$ tree-like inclusion structure of divergent subgraphs of $\mathcal{A}_{\mathcal{G},\mu}$.

TGFTs with gauge invariance condition: classification

Power-counting analysis $\Rightarrow$ **classification** of allowed interacting models:
[Oriti, Rivasseau, SC '13]

$d = \mathrm{rank}$	$D = \dim(G)$	order	explicit examples
3	3	6	$G = \mathrm{SU}(2)$ [Oriti, Rivasseau, SC '13]
3	4	4	$G = \mathrm{SU}(2) \times \mathrm{U}(1)$ [SC '14]
4	2	4	
5	1	6	$G = \mathrm{U}(1)$ [Ousmane Samary, Vignes-Tourneret '12]
6	1	4	$G = \mathrm{U}(1)$ [Ousmane Samary, Vignes-Tourneret '12]
3	2	any	
4	1	any	$G = \mathrm{U}(1)$ [Oriti, Rivasseau, SC '12]
3	1	any	

Power-counting analysis $\Rightarrow$ **classification** of allowed interacting models:
[Oriti, Rivasseau, SC '13]

$d = \mathrm{rank}$	$D = \dim(G)$	order	explicit examples
3	3	6	$G = \mathrm{SU}(2)$ [Oriti, Rivasseau, SC '13]
3	4	4	$G = \mathrm{SU}(2) \times \mathrm{U}(1)$ [SC '14]
4	2	4	
5	1	6	$G = \mathrm{U}(1)$ [Ousmane Samary, Vignes-Tourneret '12]
6	1	4	$G = \mathrm{U}(1)$ [Ousmane Samary, Vignes-Tourneret '12]
3	2	any	
4	1	any	$G = \mathrm{U}(1)$ [Oriti, Rivasseau, SC '12]
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- $d = D = 3$ with $G = \mathrm{SU}(2)$ is the only case for which a geometric interpretation is possible.
- Analogy with ordinary scalar field theory: at fixed $d = 3$
  - φ^6 model in $D = 3$;
  - φ^4 model in $D = 4$.

(ε-expansion [SC '14])

Quasi-locality of the interactions

Divergent subgraphs must be **quasi-local**, i.e. **look like trace invariants** at high scales. Always the case in known models, but non–trivial!

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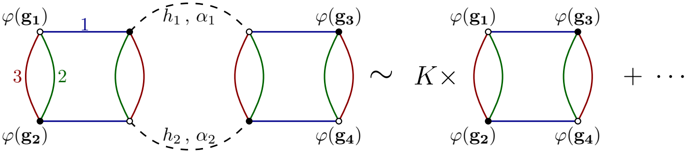

$$\int \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \int \mathrm{d}h_1 \mathrm{d}h_2 \left[K_{\alpha_1+\alpha_2}(h_1 h_2)\right]^2 \int [\prod_{i<j} \mathrm{d}g_{ij}]\, K_{\alpha_1}(g_{11} h_1 g_{31}^{-1}) K_{\alpha_2}(g_{21}^{-1} h_2 g_{41})$$

$$\delta(g_{12} g_{22}^{-1}) \delta(g_{13} g_{22}^{-1}) \delta(g_{42} g_{32}^{-1}) \delta(g_{43} g_{33}^{-1})\, \varphi(\mathbf{g_1})\, \overline{\varphi}(\mathbf{g_2})\, \overline{\varphi}(\mathbf{g_3})\, \varphi(\mathbf{g_4})$$

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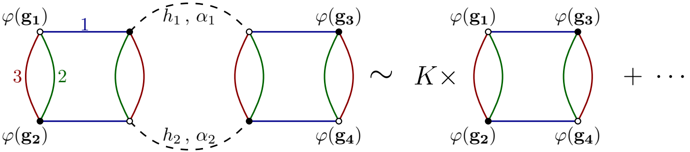

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This property is not generic in TGFTs → **"traciality" criterion**:

- flatness condition: the parallel transports must peak around $\mathbf{1}$ (up to gauge);
- combinatorial condition: connected boundary graph.

Nice interplay between structure of divergences and **topology** → renormalizable interactions are **spherical**.

Renormalized amplitudes and BPHZ theorem

Definition of renormalized amplitudes la Bogoliubov:

$$\mathcal{A}_{\mathcal{G}}^{ren} := \left(\sum_{\mathcal{F} \subset D(\mathcal{G})} \prod_{m \in \mathcal{F}} (-\tau_m) \right) \mathcal{A}_{\mathcal{G}}$$

- $D(\mathcal{G})$: set of **connected** divergent subgraphs;
- $\mathcal{F}$: **inclusion forests** of connected divergent subgraphs;
- τ_m: **contraction operator** associated to the divergent subgraph $m \rightarrow$ extracts its 'local' divergent part.

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Idea of proof:

- Use multi-scale representation of the amplitudes;
- within each $\mathcal{A}_{\mathcal{G},\mu}$, **no overlapping divergences** $\to$ finiteness from well-identified counter-terms;
- show that the sum over μ converges.

Summary and outlook

- Summary:
  - **Colored graphs** → convenient representations of pseudo-manifolds in arbitrary d.
  - **Tensor models** and **tensorial field theories** generate such colored graphs in perturbative expansion → generalizations of matrix models in arbitrary dimension.
  - **Perturbative renormalizability** well-understood, despite the complications introduced by the new notion of locality (and non-commutative group structures).
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- On-going efforts:
  - **Non-perturbative** aspects:
    - **constructive methods** [Gurau, Rivasseau '13; Lahoche '15]
    - **functional renormalization group**: Wetterich [Benedetti, Ben Geloun, Oriti '14...] and Polchinski [Krajewski, Toriumi '15] equations.
  - Hints of **non-trivial fixed points**, similar to Wilson-Fisher fixed point → **phase transitions** in quantum gravity? [Oriti '09...]
  - 4d geometric data → further constraints. Renormalizable models with Euclidean signature (group: $\mathrm{Spin}(4)$)? [Lahoche, Oriti, SC wip]
    Generalization to Lorentzian signature (group: $\mathrm{SL}(2,\mathbb{C})$): we need other methods!

Thank you for your attention