Hopf Algebras on Labelled Forests: Application to Regularity Structures

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1. Introduction
2. Labelled Trees
3. Hopf Algebras
4. Extension of the structure

Two renormalisations

The solution of a singular SPDE is described in the framework of Regularity Structures by a Taylor expansion with new monomials:

$$u(y) = u(x) + \sum_{i=1}^{N} a_i(x)\,(\Pi_x \tau_i)(y) + r(x,y)$$

where the τ_i belong to an abstract space T. We will use Hopf Algebras in order to build two groups:

- The structure group (Positive renormalisation) which defines Π_x and the map $\Gamma_{x,y}$ used for changing the point of our monomials.
- The renormalisation group (Negative renormalisation) which acts on the model $(\Pi_x, \Gamma_{x,y})$ for proving the convergence.

Hopf Algebra $\mathcal{H}$

- a product $\mathcal{M} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ satisfying:

$$\mathcal{M}\left(\mathcal{M} \otimes \mathrm{id}\right) = \mathcal{M}\left(\mathrm{id} \otimes \mathcal{M}\right), \quad \textbf{(Associativity)}$$

- a unit $\mathbf{1} \in \mathcal{H}$ satisfying:

$$\mathcal{M}(\mathbf{1} \otimes \tau) = \tau = \mathcal{M}(\tau \otimes \mathbf{1}), \forall \tau \in \mathcal{H}.$$

- a coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ satisfying:

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta, \quad \textbf{(Coassociativity)}$$

- a counit $\mathbf{1}^\star : \mathcal{H} \to \mathbb{R}$ satisfying:

$$\forall \tau \in \mathcal{H}, \left(\mathbf{1}^\star \otimes \mathrm{id}\right)\Delta\tau = \tau = \left(\mathrm{id} \otimes \mathbf{1}^\star\right)\Delta\tau.$$

Hopf Algebra $\mathcal{H}$

- The coproduct and the counit are unital algebra homomorphisms:

$$\Delta\mathcal{M} = \mathcal{M}_{\mathcal{H}\otimes\mathcal{H}}(\Delta \otimes \Delta)\,, \Delta\mathbf{1} = \mathbf{1} \otimes \mathbf{1}\,,$$
$$\mathbf{1}^\star\mathcal{M} = \mathcal{M}_{\mathbb{R}}(\mathbf{1}^\star \otimes \mathbf{1}^\star)\,, \mathbf{1}^\star(\mathbf{1}) = 1.$$

- An antipode map $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ obeying:

$$\mathcal{M}(\mathcal{A} \otimes \mathrm{id})\Delta = \mathbf{1}\mathbf{1}^\star = \mathcal{M}(\mathrm{id} \otimes \mathcal{A})\Delta.$$

Comodule and Groups

A vector space $\bar{\mathcal{H}}$ is a right comodule over $\mathcal{H}$ if there exists $\bar{\Delta} : \bar{\mathcal{H}} \to \bar{\mathcal{H}} \otimes \mathcal{H}$ such that:

$$(\bar{\Delta} \otimes \mathrm{id})\, \bar{\Delta} = (\mathrm{id} \otimes \Delta)\bar{\Delta}, \quad (\mathrm{id} \otimes \mathbf{1}^\star)\bar{\Delta} = \mathrm{id}.$$

If $\mathcal{H}^*$ denotes the dual of $\mathcal{H}$, then we set

$$G := \{g \in \mathcal{H}^* : g(\tau_1\tau_2) = g(\tau_1)g(\tau_2),\ \forall\, \tau_1, \tau_2 \in \mathcal{H}\}.$$

Theorem

Let $\mathcal{R} = \{\Gamma_g : \bar{\mathcal{H}} \to \bar{\mathcal{H}},\ \Gamma_g = (\mathrm{id} \otimes g)\bar{\Delta},\ g \in G\}$. Then $\mathcal{R}$ is a group for the composition law. Moreover, one has for $f, g \in G$:

$$\Gamma_f \Gamma_g = \Gamma_{f \circ g}, \quad f \circ g = (f \otimes g)\Delta, \quad g^{-1} = g(\mathcal{A}\cdot).$$

The polynomial structure

Take $\mathcal{H}$ the linear span of the abstract polynomials $\{X^k, k \in \mathbb{N}\}$. It is a Hopf algebra with $\mathbf{1} = X^0$ and:

- The multiplicative coproduct Δ is given by

$$\Delta X = X \otimes \mathbf{1} + \mathbf{1} \otimes X, \quad \Delta X^n = \sum_{k=0}^{n} \binom{n}{k} X^k \otimes X^{n-k}$$

- The counit $\mathbf{1}^\star$ is defined by $\mathbf{1}^\star(X^k) = \mathbf{1}_{k=0}$.
- The antipode $\mathcal{A}$ is multiplicative and given by $\mathcal{A}\mathbf{1} = \mathbf{1}$, $\mathcal{A}X = -X$.
- The structure group is isomorphic to $\mathbb{R}$ and it is given by the translation: $\Gamma_g X^k = (X + g(X))^k$

$$\Gamma_{x,y} X^k = (\Gamma_x)^{-1} \Gamma_y X^k = \Gamma_{-x}(X+y)^k = (X + y - x)^k.$$

The Wick product

We look at a very simple example of negative renormalisation: the powers of a standard gaussian r.v. ξ with zero mean and covariance $c \geq 0$.

We consider the abstract set $\mathcal{T} = \{\Xi^n : n \in \mathbb{N}\}$. Given the natural definition

$$\Pi\, \Xi^n = \xi^n,$$

we want to find M such that the renormalised n-th power of ξ is the Wick product:

$$\Pi^M \Xi^n = \xi^{\diamond n} = H_n(\xi, c)$$

where H_n are generalised Hermite polynomials. For that we consider the set $\mathcal{F} = \{\Xi^{n_1} \cdot ... \cdot \Xi^{n_k}, n_1, ..., n_k \in \mathbb{N}\}$ where the product $\cdot$ is associative and commutative with neutral $\mathbf{1} = \Xi^0$.

Renormalisation map M

We define the following coproduct $\hat{\Delta}$ by:

$$\hat{\Delta}\mathbf{1} = \mathbf{1} \otimes \mathbf{1}, \quad \hat{\Delta}\Xi^n = \sum_{n_1+\dots+n_k=n} \frac{n!}{n_1! \cdots n_k!} \Xi^{n_1} \cdot \ldots \cdot \Xi^{n_{k-1}} \otimes \Xi^{n_k}$$

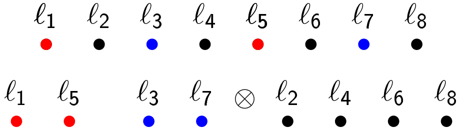
$$\hat{\Delta}(\Xi^n \cdot \Xi^m) = (\hat{\Delta}\Xi^n) \cdot (\hat{\Delta}\Xi^m)$$

with the convention that for $k = 1$, one has $\Xi^n \otimes \mathbf{1} + \mathbf{1} \otimes \Xi^n$. We consider the Hopf algebra $\mathcal{H}$ the linear span of $\mathcal{F}$. One way of defining M is $M = M_\ell$ where

$$M_\ell = (\ell \otimes id)\hat{\Delta}, \quad \ell(\Xi^n) = -c\mathbb{1}_{(n=2)}.$$

Example

We compute one term of $\hat{\Delta}\Xi^8$ associated to $A = \{\{\ell_1, \ell_5\}, \{\ell_3, \ell_7\}\}$:



Rooted Trees

We define a framework in order to treat at the same time the two renormalisations positive and negative.

A rooted tree T is a finite tree (a finite connected graph without simple cycles) with a distinguished vertex, $\rho = \rho_T$, called *the root*, and a function $\mathfrak{l}\colon L_T \sqcup E_T \to \mathfrak{L}$, where

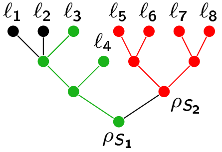
1. edges of T are denoted by $E = E_T \subset N \times N$ and nodes by $N = N_T$
2. leaves, denoted by $L = L_T$. Interior nodes, i.e. nodes which are not leaves, are denotes by $\mathring{N} = N \setminus L$.
3. We denote by $\mathbf{1}$ the (unique) labelled tree with $L = \emptyset$.
4. $\mathfrak{L}$ is a fixed non-empty set of types

Admissible Trees

Definition

Given a rooted tree T and a rooted subtree $S \subseteq T$, we say that S is *admissible in* T if

1. $L_S \subseteq L_T$
2. either $\rho_S = \rho_T$, or there exists at least one leaf $\ell \in L_T \setminus L_S$ with $\rho_S \leq \ell$.



Forests

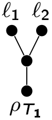
Definition

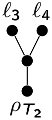
A forest F is a set of rooted trees denoted equivalently either by $\{T_1, \ldots, T_k\}$ or by $T_1 \cdot T_2 \cdots T_k$, with the empty set denoted by $\emptyset$.

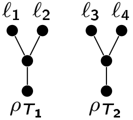
Definition

Given a forest $F = T_1 \cdot T_2 \cdots T_k$, we denote by $\mathfrak{A}(F)$ the set of all $\mathcal{A} = S_1 \cdot S_2 \cdots S_n$ such that any two elements of $\mathcal{A}$ are disjoint. Moreover for every S_i, there exists T_j such that S_i is an admissible tree in T_j.

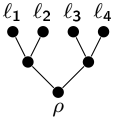
Operations on forests

Let $T_1 =$ 

and $T_2 =$ 

, then $T_1 \cdot T_2 =$ 

and

$T_1 T_2 =$ 

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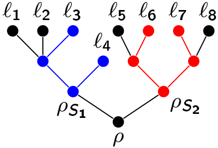
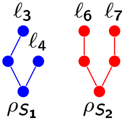
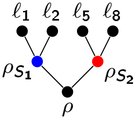
In the next example, we compute the operations of extraction-contraction for $\mathcal{A} = \{S_1, S_2\} \in \mathfrak{A}(T)$:



$\Longrightarrow \mathcal{R}^{\uparrow}_{\mathcal{A}} T =$ 

, $\mathcal{R}^{\downarrow}_{\mathcal{A}} T =$ 

Labelled trees $\mathfrak{T}$ and forests $\mathfrak{F}$

Definition

Every $T_{\mathfrak{e}}^{\mathfrak{n}} \in \mathfrak{T}$ is described with a triple $(T, \mathfrak{e}, \mathfrak{n})$ where T is a rooted tree endowed with

- an edge-labelling $\mathfrak{e} \colon E_T \to \mathbb{N}^d$
- a node-labelling $\mathfrak{n} \colon \mathring{N}_T \to \mathbb{N}^d$.

We denote by $\mathfrak{F}$ the set of labelled forests and denote a triple $(F, \mathfrak{e}, \mathfrak{n})$ by $F_{\mathfrak{e}}^{\mathfrak{n}} \in \mathfrak{F}$.

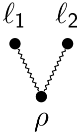
Definition

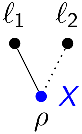
We write $\langle \mathfrak{T} \rangle$ for the linear span of $\mathfrak{T}$. We denote $\langle \mathfrak{F} \rangle$ as the free vector space generated by $\mathfrak{F}$ equipped with the product $\cdot$.

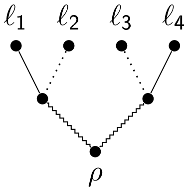
Link with the symbols

Let $T_{\mathfrak{e}}^{\mathfrak{n}} \in \mathfrak{T}$,

- To $u \in L_T$, we associate a "noise", $\Xi_{\mathfrak{l}(u)}$.
- To $e \in E_T$, we associate an abstract integrator $\mathcal{I}_{\mathfrak{e}(e)}^{\mathfrak{l}(e)}(\cdot)$.
- To $u \in \mathring{N}_T$, we associate a monomial $X^{\mathfrak{n}(u)}$.

$\mathcal{I}_1(\Xi)^2 \rightarrow$ 

$X\mathcal{I}(\Xi)\Xi \rightarrow$ 



Operations on the labels

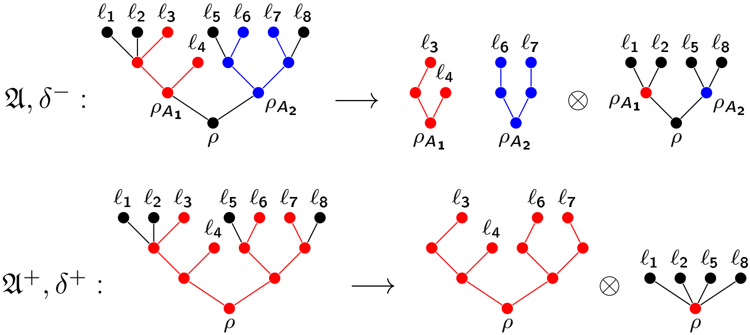
- The product in $\mathfrak{T}$ is given by $(T^{\mathfrak{n}}_{\mathfrak{e}}, \hat{T}^{\hat{\mathfrak{n}}}_{\hat{\mathfrak{e}}}) \mapsto \bar{T}^{\bar{\mathfrak{n}}}_{\bar{\mathfrak{e}}}$, $\bar{T} := T\hat{T}$ where the labels $\bar{\mathfrak{e}}$ and $\bar{\mathfrak{n}}$ are obtained by restriction except for the root, $\bar{\mathfrak{n}}(\rho_{\bar{T}}) := \mathfrak{n}(\rho_T) + \hat{\mathfrak{n}}(\rho_{\hat{T}})$.
- The product in $\mathfrak{F}$ is given by $(F^{\mathfrak{n}}_{\mathfrak{e}}, \hat{F}^{\hat{\mathfrak{n}}}_{\hat{\mathfrak{e}}}) \mapsto \bar{F}^{\bar{\mathfrak{n}}}_{\bar{\mathfrak{e}}}$, where $\bar{F} := F \cdot \hat{F}$, and labels are obtained by restriction on F and $\hat{F}$.
- The forest $\mathcal{R}^{\uparrow}_{\mathcal{A}} F^{\mathfrak{n}}_{\mathfrak{e}}$ inherits edge- and node-labels from $F^{\mathfrak{n}}_{\mathfrak{e}}$ by simple restriction.
- The forest $\mathcal{R}^{\downarrow}_{\mathcal{A}} F^{\mathfrak{n}}_{\mathfrak{e}}$ inherits the edge-labels from $F^{\mathfrak{n}}_{\mathfrak{e}}$ by simple restriction, while the node-labels are the sums of the labels over equivalence classes:

$$\mathfrak{n}([x]) \stackrel{\text{def}}{=} \sum_{y:\, y \sim_{\mathcal{A}} x} \mathfrak{n}(y).$$

Computation with $\bar{\Delta}$

Let $\bar{\mathfrak{A}} \subset \mathfrak{A}$, we define an infinite triangular linear map $\bar{\Delta}\colon \langle \mathfrak{F} \rangle \to \langle \mathfrak{F} \rangle \otimes \langle \mathfrak{F} \rangle$ by

$$\bar{\Delta} F_{\mathfrak{e}}^{\mathfrak{n}} = \sum_{\mathcal{A} \in \bar{\mathfrak{A}}(F)} \sum_{\mathfrak{n}_{\mathcal{A}}, \mathfrak{e}_{\mathcal{A}}} \frac{1}{\mathfrak{e}_{\mathcal{A}}!} \binom{\mathfrak{n}}{\mathfrak{n}_{\mathcal{A}}} \mathcal{R}_{\mathcal{A}}^{\uparrow} F_{\mathfrak{e}}^{\mathfrak{n}_{\mathcal{A}} + \pi \mathfrak{e}_{\mathcal{A}}} \otimes \mathcal{R}_{\mathcal{A}}^{\downarrow} F_{\mathfrak{e}+\mathfrak{e}_{\mathcal{A}}}^{\mathfrak{n} - \mathfrak{n}_{\mathcal{A}}}$$



Coassociativity

Theorem

If $\bar{\mathfrak{A}}$ *satisfies certain properties then the identity*

$$(\mathrm{id} \otimes \bar{\Delta})\bar{\Delta}F_{\mathfrak{e}}^{\mathfrak{n}} = (\bar{\Delta} \otimes \mathrm{id})\bar{\Delta}F_{\mathfrak{e}}^{\mathfrak{n}} ,$$

holds for every $F_{\mathfrak{e}}^{\mathfrak{n}} \in \mathfrak{F}$. *Moreover* $\bar{\Delta}$ *is multiplicative, i.e. for all* $F_1, F_2 \in \mathfrak{F}$

$$\bar{\Delta}(F_1 \cdot F_2) = (\bar{\Delta}F_1) \cdot (\bar{\Delta}F_2).$$

Finally $\bar{\Delta} \colon \langle\mathfrak{T}\rangle \to \mathfrak{F} \otimes \langle\mathfrak{T}\rangle$, *where* $\langle\mathfrak{T}\rangle$ *is the linear span of* $\mathfrak{T}$.

Homogeneity

For a labelled tree $T_{\mathfrak{e}}^{\mathfrak{n}} \in \mathfrak{T}$, we define its homogeneity by

$$|T_{\mathfrak{e}}^{\mathfrak{n}}|_{\mathfrak{s}} = \sum_{u \in L_T \sqcup E_T} |\mathfrak{l}(u)|_{\mathfrak{s}} + \sum_{x \in \mathring{N}_T} |\mathfrak{n}(x)|_{\mathfrak{s}} - \sum_{e \in E_T} |\mathfrak{e}(e)|_{\mathfrak{s}}.$$

To a symbol τ we associate a real number $|\tau|$ called its homogeneity: $|\Xi_\alpha|_{\mathfrak{s}} = |\alpha|_{\mathfrak{s}}$, $|X|_{\mathfrak{s}} = 1$, $|\mathbf{1}|_{\mathfrak{s}} = 0$

$$|\tau_1 \ldots \tau_n|_{\mathfrak{s}} = |\tau_1|_{\mathfrak{s}} + \ldots + |\tau_n|_{\mathfrak{s}}, \quad \mathcal{I}_k^{\beta}(\tau) = |\tau|_{\mathfrak{s}} + |\beta|_{\mathfrak{s}} - |k|_{\mathfrak{s}}.$$

Positive Renormalisation

Definition

A rooted tree $T \in \mathfrak{T}$ is said to be *elementary* if it either consists only of the root or has only one edge incident to the root. Let $\hat{\mathfrak{T}}$ be the corresponding set of labelled trees.

Definition

Let $\mathfrak{T}_+^0 \subset \hat{\mathfrak{T}}$ be the set of elementary labelled trees with positive homogeneity and zero label at the root. Then $\mathfrak{T}_+$ is the set of labelled trees $T_\mathfrak{e}^\mathfrak{n}$ such that $T_\mathfrak{e}^{\hat{\mathfrak{n}}}$ is a product of trees in $\mathfrak{T}_+^0$ where $\hat{\mathfrak{n}}(x) := \mathfrak{n}(x)\mathbf{1}(x \neq \rho_T)$.

Positive Renormalisation

Let $\Pi_+ : \mathfrak{T} \to \mathfrak{T}_+$ the multiplicative projection on trees with positive homogeneity. We define :

$$\Delta : \langle \mathfrak{T} \rangle \to \langle \mathfrak{T} \rangle \otimes \langle \mathfrak{T}_+ \rangle, \qquad \Delta = (\mathrm{id} \otimes \Pi_+)\delta^+$$

$$\Delta^+ : \langle \mathfrak{T}_+ \rangle \to \langle \mathfrak{T}_+ \rangle \otimes \langle \mathfrak{T}_+ \rangle, \qquad \Delta^+ = (\Pi_+ \otimes \Pi_+)\delta^+.$$

Theorem

The algebra $\langle \mathfrak{T}_+ \rangle$ endowed with the product $(\tau, \bar{\tau}) \mapsto \tau\bar{\tau}$ and the coproduct Δ^+ is a Hopf algebra. Moreover Δ turns $\langle \mathfrak{T} \rangle$ into a right comodule over $\langle \mathfrak{T}_+ \rangle$.

Positive renormalisation Group : Structure Group

We define $\mathcal{H}_+$ as $\langle \mathfrak{T}_+ \rangle$. If $\mathcal{H}_+^*$ denotes the dual of $\mathcal{H}_+$, then we set

$$G_+ := \{ g \in \mathcal{H}_+^* : g(\tau_1 \tau_2) = g(\tau_1) g(\tau_2), \ \forall \tau_1, \tau_2 \in \mathcal{H}_+ \}.$$

Theorem

Let $\mathcal{R}_+ = \{ \Gamma_g : \langle \mathfrak{T} \rangle \to \langle \mathfrak{T} \rangle, \ \Gamma_g = (\mathrm{id} \otimes g)\Delta, \ g \in G_+ \}$. Then $\mathcal{R}_+$ is a group for the composition law. Moreover, one has for $f, g \in G_+$:

$$\Gamma_f \Gamma_g = \Gamma_{f \circ g}, \quad f \circ g = (f \otimes g)\Delta^+.$$

The KPZ equation and its generalisation

The KPZ equation is given for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ by :

$$\partial_t u = \Delta u + (\partial_x u)^2 + \xi$$

where ξ is a space-time white noise.

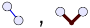
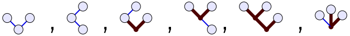
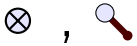
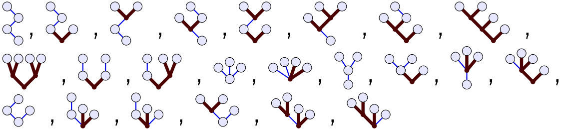
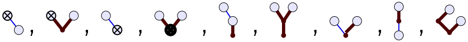
The generalized KPZ equation is given by :

$$\partial_t u = \Delta u + f(u)\,(\partial_x u)^2 + k(u)\,\partial_x u + h(u) + g(u)\,\xi.$$

- We obtain KPZ for $f \equiv g \equiv 1$ and $k \equiv h \equiv 0$.
- Contains the solution of the stochastic heat equation and invariant under composition with $C^\infty(\mathbb{R})$-functions.

Generalised KPZ

We use the following coding:  $= \Xi$,  $= \mathcal{I}(\Xi)\Xi$ and  $= \mathcal{I}_1(\Xi)^2$.

Homogeneity	Symbol(s)
$-\frac{3}{2} - \kappa$	
$-1 - 2\kappa$	
$-\frac{1}{2} - 3\kappa$	
$-\frac{1}{2} - \kappa$	
-4κ	
-2κ	
0	$\mathbb{1}$

Examples

For , the only rooted subtrees wich give positive branches are {, , , }

$$\Gamma_g = g() \circ + g() \otimes + g() + g() + g(\mathbb{1}) .$$

For , we consider the set {, , , , }

$$\Gamma_g = g() \circ + \big(g() + g() \big) + g() + g() + g(\mathbb{1}) .$$

Definition of the model $(\Pi_x, \Gamma_{x,y})$

The map Π is multiplicative and given by

$$\begin{cases} (\Pi\mathbf{1})(y) = 1, \qquad (\Pi\Xi)(y) = \xi(y), \qquad (\Pi X)(y) = y, \\ (\Pi\mathcal{I}_k\tau)(y) = \int D^k K(y-z)(\Pi\tau)(z)dz. \end{cases}$$

The map Π_x is multiplicative and given by

$$\begin{aligned} &(\Pi_x\mathbf{1})(y) = 1, \qquad (\Pi_x\Xi)(y) = \xi(y), \qquad (\Pi_x X)(y) = y - x, \\ &(\Pi_x\mathcal{I}_k\tau)(y) = \int D^k K(y-z)(\Pi_x\tau)(z)dz \\ &- \sum_{\ell=0}^{|\mathcal{I}_k(\tau)|_\mathfrak{s}} \frac{(y-x)^\ell}{\ell!} \int D^{k+\ell}K(-z)(\Pi_x\tau)(z)dz. \end{aligned}$$

Definition of the model $(\Pi_x, \Gamma_{x,y})$

The map Π_x is given by $\Pi_x = (\Pi \otimes f_x)\Delta = \Pi\Gamma_{f_x}$ where

$$f_x(\mathcal{J}_k(\tau)) = - \sum_{|\ell|_{\mathfrak{s}} < \lceil |\mathcal{I}_k(\tau)|_{\mathfrak{s}} \rceil} \frac{(-x)^\ell}{\ell!} \int D^{k+\ell} K(x-y) \Pi_x(\tau)(y) dy.$$

Then $\Gamma_{x,y} = (\Gamma_{f_x})^{-1}\Gamma_{f_y} = \Gamma_{(f_x)^{-1} \circ f_y} = \Gamma_{\gamma_{x,y}}$. In the case of smooth functions, we have a pseudo-antipode description:

$$f_x = (\Pi \mathcal{A}_+ \cdot)(x)$$

where $\mathcal{A}_+ : \mathcal{H}_+ \to \mathcal{H}$ is defined by

$$\mathcal{A}_+ \mathcal{J}_k(\tau) = - \sum_m \frac{(-X)^m}{m!} \mathcal{M}(\mathcal{I}_{k+m} \otimes \mathcal{A}_+)\Delta\tau.$$

Negative Renormalisation

Definition

Let $\mathfrak{F}_- \subset \mathfrak{F}$ be the set of all labelled forests $F_\mathfrak{e}^\mathfrak{n} = \emptyset$ or $F_\mathfrak{e}^\mathfrak{n} = (T_1 \cdot T_2 \cdots T_k)_\mathfrak{e}^\mathfrak{n}$ such that $T_i \notin \hat{\mathfrak{T}}$ and $|(T_i)_\mathfrak{e}^\mathfrak{n}|_\mathfrak{s} < 0$ for all $i = 1, \ldots, k$. The set $\mathfrak{F}_-$ is stable under the product inherited from $\mathfrak{F}$ and therefore $\langle \mathfrak{F}_- \rangle$ is an algebra.

Negative Renormalisation

Let $\Pi_- : \langle \mathfrak{F} \rangle \mapsto \langle \mathfrak{F}_- \rangle$ be the canonical projection onto $\langle \mathfrak{F}_- \rangle$. Then we define the following maps

$$\hat{\Delta} : \langle \mathfrak{F} \rangle \to \langle \mathfrak{F}_- \rangle \otimes \langle \mathfrak{F} \rangle, \qquad \hat{\Delta} = (\Pi_- \otimes \mathrm{id})\delta^-$$

$$\Delta^- : \langle \mathfrak{F}_- \rangle \to \langle \mathfrak{F}_- \rangle \otimes \langle \mathfrak{F}_- \rangle, \qquad \Delta^- = (\Pi_- \otimes \Pi_-)\delta^-.$$

Theorem

The algebra $\langle \mathfrak{F}_- \rangle$ endowed with the product $(\phi, \bar{\phi}) \mapsto \phi \cdot \bar{\phi}$ and the coproduct Δ^- is a Hopf algebra. Moreover $\hat{\Delta}$ turns $\langle \mathfrak{F} \rangle$ into a left comodule over $\langle \mathfrak{F}_- \rangle$.

Renormalisation Group

We define $\mathcal{H}_-$ as $\langle \mathfrak{F}_- \rangle$. If $\mathcal{H}_-^*$ denotes the dual of $\mathcal{H}_-$, then we set

$$G_- := \{\ell \in \mathcal{H}_-^* : \ell(\phi_1 \cdot \phi_2) = \ell(\phi_1)\ell(\phi_2),\ \forall\, \phi_1, \phi_2 \in \mathcal{H}_-\}.$$

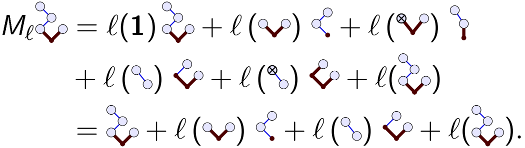
Theorem

Let $\mathcal{R}_- = \{M_\ell : \langle \mathfrak{T} \rangle \to \langle \mathfrak{T} \rangle,\ M_\ell = (\ell \otimes \mathrm{id})\hat{\Delta},\ \ell \in G_-\}$. Then $\mathcal{R}_-$ is a group for the composition law. Moreover, one has for $f, g \in G_-$:

$$M_f M_g = M_{f \circ g}, \quad f \circ g = (g \otimes f)\Delta^-.$$

Examples

For , the only subtrees which are non zero for ℓ in $\mathfrak{A}(T)$ are: $\mathbf{1}$, ,  and . Therefore, we obtain



Recursivity

One can define the map ℓ using a pseudo-antipode as for the positive renormalisation. The map ℓ is given by:

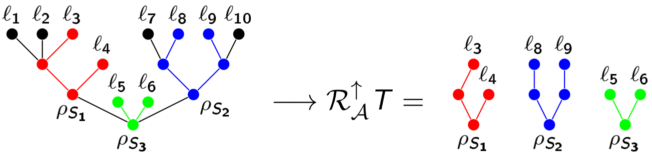
$$\ell = \mathbb{E}(\Pi \mathcal{A}_- \cdot)$$

where $\mathcal{A}_- : \mathfrak{F}_- \to \mathfrak{F}$ is defined by

$$\mathcal{A}_-(T_{\mathfrak{e}}^{\mathfrak{n}}) = - \sum_{\mathcal{A} \in \mathfrak{A}(T) \setminus \{\{T\}\}} \sum_{\mathfrak{e}_{\mathcal{A}}, \mathfrak{n}_{\mathcal{A}}} \frac{1}{\mathfrak{e}_{\mathcal{A}}!} \binom{\mathfrak{n}}{\mathfrak{n}_{\mathcal{A}}} \mathcal{A}_- \left(\Pi_- \mathcal{R}_{\mathcal{A}}^{\uparrow} T_{\mathfrak{e}}^{\mathfrak{n}_{\mathcal{A}} + \pi \mathfrak{e}_{\mathcal{A}}} \right) \mathcal{R}_{\mathcal{A}}^{\downarrow} T_{\mathfrak{e} + \mathfrak{e}_{\mathcal{A}}}^{\mathfrak{n} - \mathfrak{n}_{\mathcal{A}}, \mathfrak{n}_{\mathcal{A}} + \pi \mathfrak{e}_{\mathcal{A}}}.$$

Link between the two renormalisations

On $\mathcal{A} = \{S_1, S_2, S_3\} \in \mathfrak{A}(T)$, we compute:



Finally, we obtain $\{S_3\} \in \mathfrak{A}^+(T)$ and $\{S_1, S_2\} \in \mathfrak{A}^\circ(T)$. The set $\mathfrak{A}^\circ(T)$ as to be understood as elements of $\mathfrak{A}(T)$ without rooted subtree.

A new map δ°

With $\mathfrak{A}^\circ(T)$, we set with the usual notations the map $\delta^\circ : \langle \mathfrak{T} \rangle \mapsto \langle \mathfrak{F} \rangle \otimes \langle \mathfrak{T} \rangle$

$$\delta^\circ F_{\mathfrak{e}}^{\mathfrak{n}} := \sum_{\mathcal{A} \in \mathfrak{A}^\circ(F)} \sum_{\mathfrak{e}_{\mathcal{A}}, \mathfrak{n}_{\mathcal{A}}} \frac{1}{\mathfrak{e}_{\mathcal{A}}!} \binom{\mathfrak{n}}{\mathfrak{n}_{\mathcal{A}}} \mathcal{R}^{\uparrow}_{\mathcal{A}} F_{\mathfrak{e}}^{\mathfrak{n}_{\mathcal{A}} + \pi \mathfrak{e}_{\mathcal{A}}} \otimes \mathcal{R}^{\downarrow}_{\mathcal{A}} F_{\mathfrak{e} + \mathfrak{e}_{\mathcal{A}}}^{\mathfrak{n} - \mathfrak{n}_{\mathcal{A}}},$$

Proposition

Let $\mathcal{M} : \langle \mathfrak{F} \rangle \otimes \langle \mathfrak{F} \rangle \mapsto \langle \mathfrak{F} \rangle$, $\phi \otimes \bar{\phi} \mapsto \phi \cdot \bar{\phi}$. *Then*

$$(\mathcal{M} \otimes \mathrm{id})(\mathrm{id} \otimes \delta^\circ)\delta^+ = \delta^-.$$

Recursive Formula

For all $\ell \in G_-$, we have

$$M^\circ = M^\circ_\ell = (\ell \Pi_- \otimes \mathrm{id})\delta^\circ, \quad R = R_\ell = (\ell \Pi_- \otimes \mathrm{id})\delta^+.$$

Proposition

The map R commutes with $\mathcal{R}_+$ and one has the following recursive definition:

$$\begin{cases} M^\circ \mathbf{1} = \mathbf{1}, \qquad M^\circ X = X, \qquad M^\circ \Xi = \Xi \\ M^\circ \tau\bar{\tau} = (M^\circ \tau)(M^\circ \bar{\tau}), \qquad M\tau = M^\circ R \tau \\ M^\circ \mathcal{I}_k(\tau) = \mathcal{I}_k(M\tau) \end{cases}$$

With this recursive definition, we are able to give an expression for the renormalised model $(\Pi^M_x, \Gamma^M_{x,y})$.

Classifications of the examples

We can classify the examples according to the following properties satisfied by the model:

1. Nice: for every symbol τ, $\Pi_x^M \tau = \Pi_x M \tau$. Examples: PAM in $\mathbb{R}^2$ and the KPZ equation.
2. Medium-nice: For every symbol τ, $(\Pi_x^M \tau)(x) = (\Pi_x M \tau)(x)$. Examples: stochastic quantisation, the generalised KPZ.

The reason of not having $\Pi_x^M = \Pi_x M$

We consider the labelled tree $\bar{\tau} = \mathcal{I}(\tau)$, there exist τ_i such that $M\mathcal{I}(\tau) = \mathcal{I}(M\tau) = \mathcal{I}(\tau) + \sum_i \mathcal{I}(\tau_i)$ with $|\tau_i|_{\mathfrak{s}} > |\tau|_{\mathfrak{s}}$. Then we obtain

$$(\Pi_x^M \mathcal{I}(\tau))(y) =$$
$$\int \left(K(y-z) - \sum_{|\ell|_{\mathfrak{s}} < \lceil |\mathcal{I}(\tau)|_{\mathfrak{s}} \rceil} \frac{(y-x)^\ell}{\ell!} K(-z) \right) (\Pi_x^M \tau)(z) dz.$$

The main difference between $\Pi_x^M \bar{\tau}$ and $\Pi_x M \bar{\tau}$ is that we can have longer Taylor expansion because $|\tau_i|_{\mathfrak{s}} > |\tau|_{\mathfrak{s}}$. With $\tau = \mathcal{I}(\mathcal{I}(\mathcal{I}(\Xi)\Xi)\Xi)$, we obtain a counter-example to $\Pi_x^M = \Pi_x M$.

A new label d

For that purpose, we use the same formalism as for $\mathfrak{T}$ and we define $\mathfrak{T}_{ex}$ by:

1. We give the same meaning to the node-labels, the leaves and the edge-labels as for $\mathfrak{T}$.
2. We add a new node-label $d : N \to \mathbb{R}$ which computes a new homogeneity.

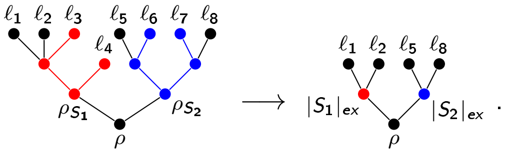
For a shape T, we denote by $T_{\mathfrak{e}}^{\mathfrak{n},d}$ such labelled tree. The new homogeneity is computed as $|T|_{ex} = |T| + \sum_{u \in N} d(u)$.

Extended the extraction-contraction operations

Let $\mathcal{A} \in \mathfrak{A}(T)$,

- we extend $\mathcal{R}^{\uparrow}_{\mathcal{A}} T$ by performing the same computation and the new node-labels is $d_{\mathcal{A}}$ the restriction of d to $\mathcal{A}$.
- we do the same for $\mathcal{R}^{\downarrow}_{\mathcal{A}} T$ and for every $A \in \mathcal{A}$, we replace $d(\rho_A)$ by $|\mathcal{R}^{\uparrow}_{A} T|_{ex}$.

In the next example, we compute $\mathcal{R}^{\downarrow}_{\mathcal{A}} T$ for $\mathcal{A} = \{S_1, S_2\}$.



The coproduct $\bar{\Delta}$

We extend the linear map $\bar{\Delta}\colon \langle\mathfrak{F}\rangle \to \langle\mathfrak{F}\rangle \otimes \langle\mathfrak{F}\rangle$ by

$$\bar{\Delta} F_{\mathfrak{e}}^{\mathfrak{n},d} = \sum_{\mathcal{A}\in\bar{\mathfrak{A}}(F)} \sum_{\mathfrak{n}_{\mathcal{A}},\mathfrak{e}_{\mathcal{A}}} \frac{1}{\mathfrak{e}_{\mathcal{A}}!}\binom{\mathfrak{n}}{\mathfrak{n}_{\mathcal{A}}} \mathcal{R}_{\mathcal{A}}^{\uparrow} F_{\mathfrak{e}}^{\mathfrak{n}_{\mathcal{A}}+\pi\mathfrak{e}_{\mathcal{A}},d} \otimes \mathcal{R}_{\mathcal{A}}^{\downarrow} F_{\mathfrak{e}+\mathfrak{e}_{\mathcal{A}}}^{\mathfrak{n}-\mathfrak{n}_{\mathcal{A}},d+\mathfrak{n}_{\mathcal{A}}+\pi\mathfrak{e}_{\mathcal{A}}}$$

Proposition

One has: $(\bar{\Delta}\otimes\mathrm{id})\bar{\Delta} = (\mathrm{id}\otimes\bar{\Delta})\bar{\Delta}$.

Trees with negative labelled d

Definition

We denote by $\mathfrak{T}^n$ the set of labelled trees with $d : N_T \to \mathbb{R}_-$. We do the same for $\mathfrak{F}^n$.

Definition

We define the positive labelled trees $\mathfrak{T}_+^n$ as the same as for $\mathfrak{T}_+$ with the new homogeneity $|\cdot|_{ex}$. We consider $\mathcal{P}_+$ the operator which sets the root label of d to 0:

$$\mathcal{P}_+ T_{\mathfrak{e}}^{\mathfrak{n},d} = T_{\mathfrak{e}}^{\mathfrak{n},\bar{d}}, \quad \bar{d} = d - \mathbb{1}_{\rho_T} d,$$

and the operator $\mathcal{P}_-$ which sets d to 0 :

$$\mathcal{P}_- T_{\mathfrak{e}}^{\mathfrak{n},d} = T_{\mathfrak{e}}^{\mathfrak{n},0} = T_{\mathfrak{e}}^{\mathfrak{n}}.$$

Coproduct

Let $\Pi_+ : \langle \mathfrak{F}^n \rangle \mapsto \langle \mathfrak{T}^n_+ \rangle$, $\Pi_- : \langle \mathfrak{F} \rangle \mapsto \langle \mathfrak{F}_- \rangle$ be the canonical projection onto $\langle \mathfrak{T}^n_+ \rangle$, resp. $\langle \mathfrak{F}_- \rangle$. Then we define the following maps

$$\Delta : \langle \mathfrak{T}^n \rangle \to \langle \mathfrak{T}^n \rangle \otimes \langle \mathfrak{T}^n_+ \rangle, \qquad \Delta = (\mathrm{id} \otimes \Pi_+ \mathcal{P}_+)\delta^+$$

$$\Delta^+ : \langle \mathfrak{T}^n_+ \rangle \to \langle \mathfrak{T}^n_+ \rangle \otimes \langle \mathfrak{T}^n_+ \rangle, \qquad \Delta^+ = (\Pi_+ \mathcal{P}_+ \otimes \Pi_+ \mathcal{P}_+)\delta^+$$

$$\hat{\Delta} : \langle \mathfrak{F}^n \rangle \to \langle \mathfrak{F}_- \rangle \otimes \langle \mathfrak{F}^n \rangle, \qquad \hat{\Delta} = (\Pi_- \mathcal{P}_- \otimes \mathrm{id})\delta^-$$

$$\Delta^- : \langle \mathfrak{F}_- \rangle \to \langle \mathfrak{F}_- \rangle \otimes \langle \mathfrak{F}_- \rangle, \qquad \Delta^- = (\Pi_- \mathcal{P}_- \otimes \Pi_- \mathcal{P}_-)\delta^-.$$

Hopf Algebras

Theorem

The algebra $\langle \mathfrak{T}_+^n \rangle$ endowed with the product $(\tau, \bar{\tau}) \mapsto \tau\bar{\tau}$ and the coproduct Δ^+ is a Hopf algebra. Moreover Δ turns $\langle \mathfrak{T}^n \rangle$ into a right comodule over $\langle \mathfrak{T}_+^n \rangle$.

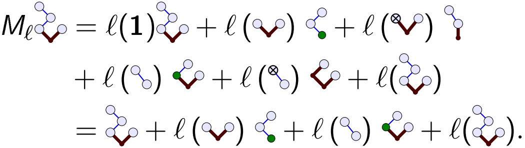
Theorem

The algebra $\langle \mathfrak{F}_- \rangle$ endowed with the product $(\phi, \bar{\phi}) \mapsto \phi \cdot \bar{\phi}$ and the coproduct Δ^- is a Hopf algebra. Moreover $\hat{\Delta}$ turns $\langle \mathfrak{F}^n \rangle$ into a left comodule over $\langle \mathfrak{F}_- \rangle$.

The group $\mathcal{R}_-$ is unchanged whereas the group $\mathcal{R}_+$ takes into account the new label d.

Examples

For , the only subtrees which are non zero for ℓ in $\mathfrak{A}(T)$ are: $\mathbf{1}$, ,  and . Therefore, we obtain



New property

Let Δ° defined by

$$\Delta^\circ : \langle \mathfrak{T}^n \rangle \mapsto \langle \mathfrak{F}_- \rangle \otimes \langle \mathfrak{T}^n \rangle, \qquad \Delta^\circ = (\Pi_- \mathcal{P}_- \otimes \mathrm{id})\delta^\circ.$$

Proposition

One has the following identities:

$$\mathcal{M}^{(13)(2)(4)} \left(\Delta^\circ \otimes \Delta^\circ\right) \Delta = (\mathrm{id} \otimes \Delta)\Delta^\circ$$
$$\mathcal{M}^{(13)(2)(4)}(\hat{\Delta} \otimes \Delta^\circ)\Delta = (\mathrm{id} \otimes \Delta)\hat{\Delta}.$$

where $\mathcal{M}^{(13)(2)(4)}$ *is defined by*

$$\mathcal{M}^{(13)(2)(4)}(\tau_1 \otimes \tau_2 \otimes \tau_3 \otimes \tau_4) = (\tau_1 \cdot \tau_3 \otimes \tau_2 \otimes \tau_4).$$

Nice identity

Proposition

Let $T^{\mathfrak{n}}_{\mathfrak{e}}$ *and* $\ell \in G_-$, *then* $\Pi^{M_\ell}_x = \Pi_x M_\ell$, $\gamma^{M_\ell}_{x,y} = \gamma_{x,y} M^\circ_\ell$ *and*

$$\Pi^{M_\ell}_x T^{\mathfrak{n}}_{\mathfrak{e}} = \sum_{\mathcal{A} \in \mathfrak{A}(T)} \sum_{\mathfrak{e}_\mathcal{A}, \mathfrak{n}_\mathcal{A}} \frac{1}{\mathfrak{e}_\mathcal{A}!} \binom{\mathfrak{n}}{\mathfrak{n}_\mathcal{A}} \ell \left(\Pi_- \mathcal{R}^\uparrow_\mathcal{A} T^{\mathfrak{n}_\mathcal{A} + \pi \mathfrak{e}_\mathcal{A}}_{\mathfrak{e}} \right)$$
$$\Pi_x \left(\mathcal{R}^\downarrow_\mathcal{A} T^{\mathfrak{n} - \mathfrak{n}_\mathcal{A}, \mathfrak{n}_\mathcal{A} + \pi \mathfrak{e}_\mathcal{A}}_{\mathfrak{e} + \mathfrak{e}_\mathcal{A}} \right)$$

$$\Pi^{M_\ell}_x T^{\mathfrak{n}}_{\mathfrak{e}} = \left(\mathbb{E}(\Pi \mathcal{A}_- \cdot) \otimes \Pi \otimes (\Pi \mathcal{A}_+ \cdot)(x) \right) (\mathrm{id} \otimes \Delta) \hat{\Delta} T^{\mathfrak{n}}_{\mathfrak{e}}.$$