The weak KPZ universality conjecture in equilibrium

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Paths to, from and in renormalization
Potsdam

Based on joint works with Joscha Diehl and Massimiliano Gubinelli

Motivation: modelling interface growth

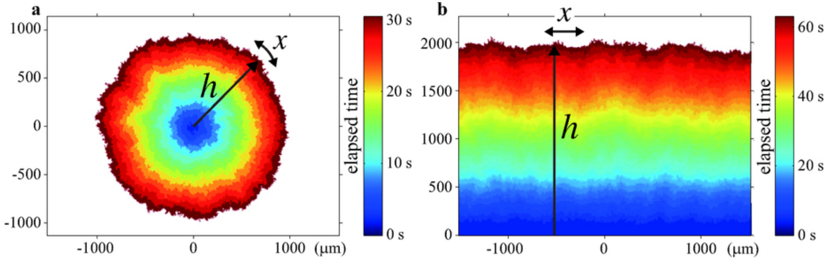

Figure: Takeuchi, Sano, Sasamoto, Spohn (2011, Sci. Rep.)

KPZ equation

- Model for (fluctuations in) random interface growth: $h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

$$\partial_t h(t,x) = \underbrace{\Delta h(t,x)}_{\text{diffusion}} + \underbrace{\lambda |\partial_x h(t,x)|^2}_{\text{slope-dependence}} + \underbrace{\xi(t,x)}_{\text{space-time white noise}}$$

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- Kardar-Parisi-Zhang (1986): slope-dependent growth $F(\partial_x h)$;

$$F(\partial_x h) = F(0) + F'(0)\partial_x h + \frac{1}{2}F''(0)(\partial_x h)^2 + \dots$$

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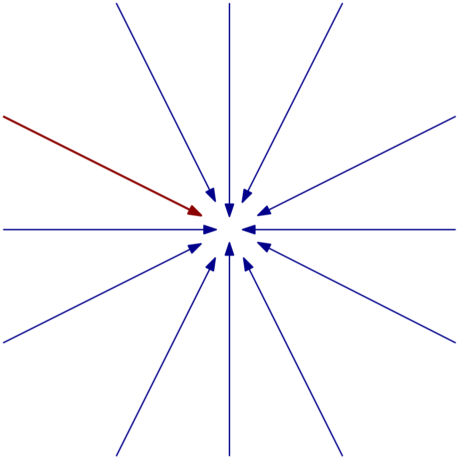
  Highly non-rigorous since $\partial_x h$ is a distribution; come back to this later.

- Conjecture: fluctuations

$$\varepsilon^1(h(t\varepsilon^{-3}, x\varepsilon^{-2}) - \varphi(t,x))$$

  converge to KPZ fixed point. Only known for fixed t, special h_0 (Amir et al. (2011), Sasamoto-Spohn (2010), Borodin et al. (2014)). Difficulty: KPZ fixed point is very poorly understood.

KPZ universality conjecture

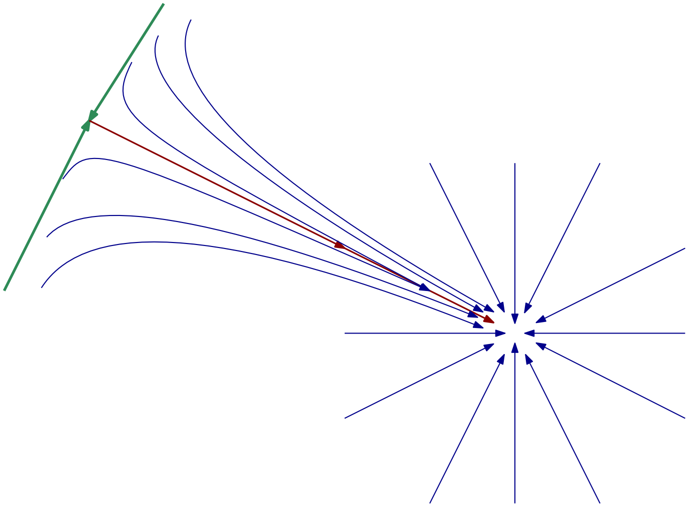


All interface models converge to KPZ fixed point under "renormalization"

$$f \mapsto \varepsilon f(\cdot\varepsilon^{-3}, \cdot\varepsilon^{-2}).$$

Red line: KPZ equation

Weak KPZ universality conjecture: physics



Gaussian fixed point and KPZ fixed point connected by one-dimensional curve: $\partial_t h = \Delta h + \lambda |\partial_x h|^2 + \xi$.

Weak KPZ universality conjecture: mathematics

Consider class of stochastic models (f^ε) on $[0,\infty) \times \mathbb{R}$ or $[0,\infty) \times \mathbb{Z}$ with

- exactly one conservation law;
- tunable strength of asymmetry ε.

Then $\exists$ observable u^ε of f^ε such that

$$\varepsilon^{-1}(u^\varepsilon(t\varepsilon^{-4}, x\varepsilon^{-2}) - \varphi(t,x)) \Rightarrow u(t,x),$$

where $u = \partial_x h$ solves Burgers equation

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- u^ε measures interface height differences at neighboring sites;
- can go back and forth between h and $u = \partial_x h$, but u mathematically more convenient.

Weak KPZ universality conjecture: examples

Simple exclusion process:

- particles on $\mathbb{Z}$ move independently, jump left with rate p, right with rate $1-p$;
- jump suppressed if landing site occupied;
- conservation law: no. of particles;
- $p=\frac{1}{2}$: convergence to Gaussian; $p=\frac{1}{2}+\varepsilon$: expect $\Rightarrow$ Burgers.

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Ginzburg-Landau $\nabla\varphi$ model:

- Interacting Brownian motions on $\mathbb{Z}$:

$$\mathrm{d}x^j = \big(pV'(r^{j+1}) - (1-p)V'(r^j)\big)\,\mathrm{d}t + \mathrm{d}w^j; \quad r^j = x^j - x^{j-1};$$

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Hairer-Quastel model:

- Replace quadratic nonlinearity in KPZ by general one, smooth noise:

$$\partial_t v = \Delta v + \alpha F(\partial_x v) + \rho * \xi$$

- $\alpha = 0$: (convergence to) Gaussian; $\alpha=\varepsilon$: expect $\Rightarrow$ KPZ.

Weak KPZ universality conjecture: metatheorem

Metatheorem (Gonçalves-Jara '13, Gubinelli-Jara '13, Gubinelli-P. '15)

We have a set of tools allowing to verify the weak KPZ universality conjecture for a large class of models starting from equilibrium.

- *Go-Ja '13:* *Tools for tightness and martingale characterization of limits;*
- *Gu-Ja '13:* *refined martingale characterization;*
- *Gu-Pe '15:* *uniqueness of refined martingale characterization.*

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- *Gu-Pe '15: uniqueness of refined martingale characterization.*

Tested on many models:

- Generalizations of simple exclusion process: Gonçalves-Jara (2014), Gonçalves-Jara-Simon (2016), Franco-Gonçalves-Simon (2016);
- Zero-range process and many other interacting particle systems: Gonçalves-Jara-Sethuraman (2015);
- Ginzburg-Landau $\nabla\varphi$ model: Diehl-Gubinelli-P. (2016, unpublished);
- Hairer-Quastel model: Gubinelli-P. (2016).

How to solve KPZ

$$\partial_t h = \Delta h + |\partial_x h|^2 - \infty + \xi.$$

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$$\partial_t w = \Delta w + w\xi$$

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- Hairer (2013): series expansion and rough paths/regularity structures, defines $|\partial_x h|^2 - \infty$.
- Martingale problem: Assing (2002), Gonçalves-Jara (2010/2014), Gubinelli-Jara (2013) define "energy solutions" of equilibrium KPZ. Uniqueness long open, solved in Gubinelli-P. (2015).

Different solutions and weak universality

- Difficulty with Cole-Hopf: most systems behave badly under exp-transform, only ok for few specific models: Bertini-Giacomin (1997), Dembo-Tsai (2013), Corwin-Tsai (2015), Corwin-Shen-Tsai (2016).

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- Martingale problem: most powerful tool for universality if invariant measure known explicitly (all the works mentioned above). Approach:
  -Show tightness of fluctuations;
  -Give martingale characterization of limit points;
  -Uniqueness $\Rightarrow$ convergence.

Conclusion

- KPZ universality conjecture:
  asymmetric growth models $\Rightarrow$ KPZ fixed point.
- Weak KPZ universality conjecture:
  weakly asymmetric growth models $\Rightarrow$ KPZ equation.
- Both difficult to show because conjectured limits are difficult objects.
- But with recent breakthroughs on KPZ equation: weak conjecture becomes tractable.
- In particular with martingale approach: good handle on equilibrium situation, can establish conjecture in that case.

Thank you